WORKING GROUP 15

THE ROLE OF HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION: THEORY AND RESEARCH

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Index of papers and posters to be presented

THE TEACHING OF VECTORS IN MATHEMATICS AND PHYSICS IN FRANCE DURING THE 20TH CENTURY

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The work presented in this text is part of a doctorial dissertation in mathematics education (Ba 2006) about the teaching and learning of vectors, translations, forces, velocity and movement of translation in mathematics and physics. Here, we present the evolution of the teaching of vectors and vector quantities in mathematics and physics from the end of the 19th century up to now. We analyse this evolution in the light of the ecology of knowledge, as developed by Yves Chevallard (1994). This helps us understand the difficulties in recent periods, in order to create a successful interdisciplinary approach in the teaching of these notions in mathematics and physics.

INTRODUCTION

Vectors emerged during the 19th century at the border of mathematics and physics. We will not recall here their historical evolution (see e. g. CROWE 1967, DORIER 1997 and 2000, FLAMENT 1997 and 2003). Our interest is clearly into the history of their teaching in the curricula of both mathematics and physics in France since the end of the 19th century. Today, in France, vectors in mathematics occupy a small part of the curriculum of geometry in secondary education (8th to 12th grades), while vector quantities are taught in Physics in 11th and 12th grades. Introducing an interdisciplinary approach has been suggested in recent programs, but is yet not very successful, as shown by our study of textbooks and teachers' practices (BA 2006, BA & DORIER 2007). The bad effects of partitioning in curricula between mathematics and physics teaching has been pointed out, especially about vectors, by several authors (see LOUNIS 1989 for a review). In this context, our aim is to understand how such a partitioning has been made possible, in order to find a way to make the interrelation between mathematics and physics teaching better.

The ecological approach developed by CHEVALLARD (1994), is a theoretical tool proper to help us tackle this issue. Indeed, it allows to study the different positions and functions of vectors and vector quantities in the moving landscape of mathematics and physics teaching, with conditions and constraints for survival and development. The idea is to analyse the evolution of objects of knowledge in various (didactic) institutions like organisms in various ecosystems.

> The ecologists distinguish, when referring to an organism, its habitat and its niche. To put it in an anthropomorphic way, the habitat is, in a way, the address, the place where it lives. The niche regroups the functions that the organism fulfils. It is, in a way, its profession in this habitat[1]. (Op. cit., p. 142).

Following CHEVALLARD, ARTAUD (1997) analyses under which conditions new objects can emerge and live in an ecosystem.

> For a new object of knowledge O to emerge in a didactical ecosystem, it is necessary that a certain milieu exists for this object, i.e. a set of known objects (in the sense that a non problematic institutional relation exists) with which O comes in interrelation. […] A mathematical object cannot exist on its own; it must be able to occupy a specific position in a mathematical organisation, that has to be brought to life. The necessity for a milieu implies that a new mathematical organisation cannot emerge ex nihilo. It must lean on already existing mathematical or non-mathematical organisations[i]. (Op. cit., p. 124).

The ecological approach consists therefore in bringing to light a network of conditions and constraints that determines the evolution of the positions that objects (vectors in our work) can have in the different periods corresponding to changes in the programs. In this perspective, we have to take into account various institutions (and their specific constraints): school in general, but also mathematicians and physicists.

We do it chronologically from 1852 up to today, according to various phases, corresponding to the main teaching reforms.

THE BEGINNINGS (1852-1925)

In 1852, techniques for obtaining the resultant of two forces is taught in physics in 11th grade (age 17). There is a reference to the parallelogram of forces, but no vectors as such, just a technique based on a geometrical pattern. The same year the term radius vector (*rayon vecteur*) is used in geometry. This comes from astronomy, where the radius vector designates the segment joining one of the foci of the ellipse describing a planet's trajectory to its position on the orbit. It has therefore not much to do with what we call a vector now.

Until 1902, vector and vector quantities are absent from French secondary teaching both in mathematics and physics. In 1902, the radius vector disappears, but the vector, as a directed line segment appears in the program of 11th grade in mechanics and kinematics, part of mathematics then. Meanwhile, in 9th grade too, in statics and dynamics, the scalar product is used to calculate the work done by a force. Therefore vectors enter the curriculum in 11th grade in the habitat of what we can call "paraphysics"[2], with a niche as representations of orientated quantities. This is coherent with their origin and use in science of that time. It is also coherent with the general aims of the 1902 reform, which promotes mathematics as the root of natural sciences. Moreover, the 1902 reform insists on collaborations between mathematics and physics teachers:

> It would be good that [...] mathematics and physics teachers in the same school support each other mutually. Physics teachers must always know at what stage of mathematics knowledge are their students and conversely mathematics teachers would gain in not ignoring some examples that they could choose, in the experimental knowledge already acquired, in order to illustrate the theories they have explained in an abstract way. (Introduction to Programmes du lycée, 1902, p.3)

The 1902 reform is quite ambitious and gives to the sciences and mathematics in particular a privileged position. A result of this ambition is that the curriculum is too important, therefore teachers complain that it is impossible to cover everything. In 1905, the ministry of education has to reduce the program. In this technical adaptation, vectors are moved from 11th to 12th grade and enter a new habitat, since they are now part of the geometry curriculum, where they have to be presented as tools for physics (their niche):

> In mechanics, […] teachers must avoid any development on purely geometrical aspect; it is in order to suppress any such occasion, that theorems on vectors have been reduced to a minimum and moved in the geometry curriculum, where they appear under their real aspect[i]. (Instruction du 27 juillet 1905 relative à l'enseignement des mathématiques, p. 676)

Vectors are therefore transported from mathematical physics into geometry, in order to technically solve a purely didactical problem.

In 1925, without being explicitly in the program, vectors appear in the 9th grade, as a possible concrete representation of "algebraic numbers", "concrete notions on positive and negative numbers". This is a new potential habitat in arithmetics, as representations of one-dimensional orientated quantities (their niche). Here again, the reasons are mostly of didactical order.

In 12th grade, the content about vectors remains more or less the same than during the preceding period. Yet, vectors have migrated into trigonometry, for which they facilitate the didactical presentation. In kinematics, the use of vectors to represent velocity and acceleration is more systematic, like in mechanics, with forces. The habitat and niche in physics are therefore reinforced. Meanwhile, a comment in the program in 1925 is quite interesting:

> In statics, the confusion that happened very often between the properties of systems of forces and those of associated systems of vectors, will disappear because of the general study of the latter.

Therefore the geometrical status of vector is reinforced, so is their niche in this habitat, due to the new connection with trigonometry.

In a bit more than 20 years, fore purely didactical reasons, vectors initially hybrid objects at the border between physics and mathematics, acquired a geometrical status and a potential arithmetical one. Their use in physics is not anymore essential, since they have to be introduced separately.

A SLOW EVOLUTION (1937-1967)

In 1937, the use of vectors to represent algebraic numbers in 9th grade is made official, and the projection of parallel vectors on the same axis is suggested as a means to illustrate the multiplication of numbers with a sign. In the same vein, vectors are used in the presentation of homotheties. The arithmetical habitat is therefore reinforced.

The habitat in trigonometry remains but is moved down to 11th grade.

Habitats and niches are therefore identical. Clearly one-dimensional vectors live in arithmetic for the 9th grade, where multiplication by a scalar is important, while higher dimensional vectors are introduced in the 11th grade in trigonometry. The habitat in physics appears later, but more systematically, as an application. No mention of possible bridges between the different habitats is made, while difficulties in the use of vectors in physics are noticed officially.

In 1947, there are no major changes. For the first time, vectors are used to present a vector version of Thales' theorem in the 9th grade, following the use of vectors for homotheties. In the 11th grade, vectors are now a separate chapter in geometry, no longer part of trigonometry. The term of equipollent vector is introduced, and the link with translation is made.

Therefore, vectors have now gained an autonomous mathematical status. The dichotomy between arithmetics (one dimension) and geometry (higher dimension) still exists. Yet, Thales' theorem makes a bridge between the two habitats, and put forward the multiplication by a scalar, which originally was not very important in the use for physics.

In 1957, the potential bridge between the arithmetical and geometrical habitat is made. Vectors appear in the 9th grade, in geometry, in relation with proportional transformations and Thales' theorem: the arithmetic habitat has been absorbed into geometry. In the 10th grade, 3 dimensional directed line segments are introduced as part of the geometry curriculum, in relation with translations and analytic geometry. In the 11th grade, the distinction between directed line segments and free vectors is made. Applications to geometry and kinematics are important. Barycentres also appear for the first time and are linked to vectors. The geometric habitat is therefore stronger and has absorbed the arithmetic habitat, which only survive in a transitory phase in the 9th grade. In this enlarged geometric habitat, the niche is not anymore the representation of vector quantities from physics, but more an efficient tool for solving geometrical problems. For educational purposes, vectors have therefore become geometrical objects. They are used to introduce analytic geometry and barycentres, two fields of geometry that historically existed before vectors!

In physics, in 12th grade, vectors are also used in magnetism, yet mostly through representation by coordinates. This, again, is quite ironical, compared to the historical development, when one recalls that Maxwell's formulae played an important role in the history of vectors, to impose the coordinate-free notations!

MODERN MATHEMATICS (1968-1985)

In the enormous changes brought by modern mathematics, geometry teaching was to be profoundly renewed. Vectors were introduced in 7th grade, very formally. In 9th grade, the axiomatic structure of vector space was defined, yet limited to finite dimensions. In his history of linear algebra, Dorier (1997 or 2000a) has shown that the model of geometrical space, as the Euclidean three-dimensional vector space has been promoted by Dieudonné (1964) because, in his mind, it was the best preparation for the Hilbert and more general function spaces, which were important in the curriculum for post graduates in mathematics. Indeed, promoters of modern mathematics (among whom Dieudonné was one of the most radical) had a descending view of mathematics education: students had to be trained as young as possible to ideas that were essential to professional mathematicians. In this perspective, introducing geometry through vectors made possible to introduce the structure of Euclidean vector space very early. "Geometrical vectors" became then the (quasi unique) prototype of Euclidean vector spaces. Yet, this is a reduction and a deviation from the historical genesis.

> […] the nature of the geometrical vector […] is the outcome of a dialectical perspective between algebraic structure and geometric intuition. It has to be underlined here that the expression "algebraic structure" does not mean that the geometrical vector is essentially the emergence of the theory of vector space in geometry. Indeed, one should not be misled by the proximity of vocabulary. The theory of vector space is by nature axiomatic, algebraic vectors (elements of a vector space) are not constructed, they are given objects defined only by their properties as element of a structure. Geometrical vectors on the contrary are the result of a dynamic process of abstraction: the object is created through an algebraic elaboration in interaction with geometric intuition. Moreover, the roles of vector and scalar products have been essential in the genesis of geometrical vector, whereas the linear structure put forward the multiplication by a scalar, which is not essential with regard to geometrical vectors[i]. (DORIER 2000b, pp. 76-77)

A totally new mathematical organisation took place in geometry, in which vectors were central. But the nature of vectors was also changed, they became mostly examples of linear algebra theory. Therefore, a new niche appeared in the habitat of geometry: preparation of students to linear algebra, which was taught from 10th grade, up to post-graduate level (functional analysis). Vectors were also used in Physics, but the gap between formal objects and applications got very important and many students had difficulties:

> The coordination mathematics-physics is getting complicated: in addition to the time lag between mathematics teaching and the needs of physics teaching there is a gap between modern mathematics taught and applicable mathematics used in the teaching of physics.

> Thus, a group will be constituted at the junction between the Laguarrigue and the Lichnerowicz commissions[3].[i] (BELHOSTE, GISPERT & HULIN 1996, p. 112)

Research works in physics education in the seventies pointed out several difficulties in the use of mathematics in physics, especially regarding vectors. MALGRANGE, SALTIEL & VIENNOT (1973) for instance interviewed students entering university and pointed out that a correct use of addition of vectors about forces or velocities was a major problem.

However, it is well known that the reform was quickly criticised and rejected.

> A reform conducted by tertiary education for its own sake and interest without any clear vision of missions specific to secondary education, was certainly bound to fail right from the beginning, whatever was its scientific legitimacy and its promoters' good will. (BELHOSTE, GISPERT & HULIN 1996, p. 37)

In the late seventies, some modifications were adopted, but it is only in the early eighties, that a total reconstruction of the curricula took place.

THE COUNTER REFORM (1985-2002)

Following the failure of introduction of modern mathematics, in 1985 the teaching of vector space theory disappears from secondary education, replaced by a more concrete approach to geometry. The new program specifies: "vectors should not be only algebraic entities; mastering their relations with configurations play an essential role in the solving of geometric problems".

> This eludes the fact that vectors are intrinsically algebraic, and that this algebraic nature does not refer just to the theory of vector space. Operations on geometrical vectors are part of their constitution as objects :
>
> - Magnitude is the basis of arithmetic since Ancient Greeks.
> - Orientation on the same line is what allows considering negative entities, a decisive step towards addition.
> - Direction finally comes from the necessity of multiplication.
>
> This last idea is the most complicated to understand. But, let us look at what is vector multiplication. In Greek algebraic geometry, the product of two numbers (lines) is the rectangle's area. If one considers a parallelogram instead of a rectangle, the sine of the angle formed by the two lines has to be taken into account in the formula for the area, i.e. the relative position of the two lines (the idea of negative implies to take into account the orientation of the lines). Thus, like Grassmann (1844) underlines it, in his introduction to the *Ausdehnungslehre*, the parallelogram, not the rectangle, symbolises the true concept of multiplication, if one considers orientated entities in geometry. This brings to light the importance of direction of lines in the construction of the product[i]. (DORIER 2000b, pp. 79-80).

As a consequence of the rejection of any formal viewpoint in the teaching of vectors, these appear as tools for solving geometric problems, and eventually for physics, but have no clear status as objects. Even the use of vectors to illustrate operation on one-dimensional orientated quantities has disappeared. After the rejection of modern mathematics, the teaching of vector is lacking of theoretical reference. The model of linear algebra has been banished but nothing came in the place. Yet, some residues remain in few places. For instance it is still common today in textbooks for 10th grade, to show that vectors have some properties, which are actually the axioms of vector space (but it is not explicit).

Since the counter-reform in France, vectors are introduced in a naïve way in relation with translation. This viewpoint is not new, it has been developed for instance by Jacques HADAMARD (1898) in his *Leçons de géométrie:*

> If by all the points of a figure, one draws equal parallel lines with the same orientation, the end points of these lines constitutes a figure equal to the original. […] The operation through which one passes from the first to the second figure was given the name of translation. One sees that a translation is determined when a line is given in magnitude, direction and orientation such as *AA'*, which goes from one point to its homologue. Thus a translation is designated by the letter of such a line: e.g. the translation *AA'*[i]. (op. cit., p. 51).

The vector first introduced in the 8th grade, finally got introduced only in the 9th grade. Moreover, in recent years, the content about vectors has been reduced to a minimum. The link with physics is promoted in the programs. But, as our survey of textbooks and teachers' practices (BA 2007) showed, it is very limited and very often not effective. On the other hand, vectors are used in physics to represent forces and velocity, but physics teachers keep complaining that their students are not competent enough with vectors.

In this last period, the habitat of vectors has been reduced to a small part in geometry. They are presented as efficient tools to solve geometric problems and models for forces and velocity. These niches however have difficulty in surviving. Indeed, several research works in mathematics education (e.g. BITTAR 1998, LE THI HOAI 1997, PRESSIAT 1999) have shown the difficulty in convincing students of the power of vectors for solving geometric problems. On the other hand, the distance and partitioning between mathematics and physics teaching makes the interrelation difficult. In our work, we have studied this problem not only about vectors but also about translations and movement of translation (BA & DORIER 2007).

CONCLUSION

Despite the rejection of modern mathematics in the eighties, the model of linear algebra, even if it has disappeared from secondary education, remains implicitly the only algebraic model for vectors, influencing the mathematical organisation of the teaching of vectors. In this sense, the multiplication by a scalar is overestimated while, on the contrary, the vector product is underestimated. The axioms of vector spaces appear implicitly, while algebraic aspects more specific to geometric vectors are eluded, like the link with Thales' theorem and one-dimensional orientated quantities. The vanishing of any algebraic habitat or niche is like something missing after the (well founded) rejection of linear algebra. A reflection on the true algebraic nature of geometric vector and its link with geometric intuition is totally absent of the teaching of vector, since the beginning, while it had been an essential aspect in the genesis of vectors.

The niche "efficient tool for solving geometric problems" is quite problematic. It is indeed difficult to find geometric problems, accessible to students in 10th grade, in which vectors appear really as more efficient than more basic geometric methods. Moreover, our study of the evolution of the teaching of vectors shows that the geometric habitat was not "natural" at the beginning. From its origin as hybrid objects between mathematics and physics, vectors have been transformed, in a didactical process of transposition, into geometric entities. We have shown that several changes between 1925 and the beginning of modern mathematics have been motivated by purely didactical (not epistemological) constraints. Ideology on teaching and practical reasons often (if not always) have surpassed scientific motives. The changes occurred during the reform of modern mathematics are even more obviously driven by ideology and subject to suspicion on epistemological grounds.

The niche "tool for physics' entities" remains throughout the century up to now. Yet, our analysis of the evolution of the teaching of vectors shows that the gap between habitats in mathematics and in physics has constantly grown bigger. Until the sixties, parts of mechanics and kinematics constituted a common ground between mathematics and physics where vectors were used. Even then, an artificial separation was made and vectors got "rejected" in geometry. In today's mathematics textbooks, the examples taken from physics to illustrate the use of vectors are mostly inaccurate and often wrong from a physicist's viewpoint, while physics teachers refuse to do mathematics and expect mathematical tools to be at disposal in time (BA & DORIER in press).

For the interrelations between mathematics and physics teaching to get better, changes in the curricula will be necessary, but it will not be sufficient. For each subject capable of strengthening the relations between mathematics and physics, an epistemological analysis has to be conducted in order to make the adequate changes. Our claim is that this study must take into account the historical evolution of the concepts at stake AND the evolution of the teaching of these concepts, with a description of the constraints of the educational context. Such analyses must be the bases for teaching experimented completed by didactical analysis. Finally specific teachers' training is necessary, in order to make the changes possible.

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[1] Our translation.

[2] This designates the topics at the border between physics and mathematics, a border that moved along the time and according to different countries.

[3] The official commissions in charge of designing the new teaching respectively in physics and mathematics.

GEOMETRY TEACHING IN ICELAND IN THE LATE 1800S AND THE VAN HIELE THEORY

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The main issue of the paper is the first Icelandic textbook in geometry, published in 1889, and its declared aim to avoid formal proofs. In the late 19th century, there were discussions in Europe about geometry instruction; if it should be taught as purely deductive science or built on experimental and intuitive thinking. Icelandic intellectuals stayed outside mainstreams of philosophical and didactic discussions, while their policy was to enhance strategies to lead their country towards independence and technical progress. In the paper these discussions are connected to the van Hiele theory on geometric thinking.

INTRODUCTION

Iceland has a well recorded history of its educational and cultural issues since its settlement around 900 AD. A great collection of literature of various kinds exists from the 12th–14th century. Among that is literature of encyclopaedic nature, which contains some mathematics, mainly arithmetic and chronology. There is little evidence that geometry of the *Elements* was ever studied in the two cathedral schools in Iceland in the period from the 12th to the early 19th century, while astronomical observations and geodetic measurements were made in the 1500s, 1600s and 1700s by local people who had studied at Northern-European universities.

Iceland became a part of the Danish Realm by the end of the 14th century. The two cathedral schools were united into one state school in 1802. Their goal was to prepare their pupils for priesthood and for studies at the University of Copenhagen.

From the middle of the 19th century there were increased demands for independence from Denmark. Detailed proposals were written on schools for farmers and a lower secondary school for the middle class as means towards raising educational standards of a future independent nation. Classical geometry was to be provided for those heading for university entrance, while practical measuring skills and geodesy were proposed for future farmers.

As a milestone towards independence, the Icelandic parliament became a legislative body in 1874; an event followed up with legislation in 1880 on teaching children arithmetic and writing, which was, as reading since the 1740s, the responsibility of the families under the supervision of the parish priests until the early 20th century.

Another milestone was a public lower secondary school, run by the state, established in 1880 in Northern-Iceland. The school was intended for future farmers and craftsmen, while in 1908 its final examination sufficed for entrance into the Reykjavík School. Its syllabus thus became more theoretic with time. It remained the only school of its kind until 1928. Several privately run lower secondary schools as well as technical schools were established from the 1880s with some support from the state, increasing in number after the turn of the century.

Along with the establishment of schools, textbooks in the vernacular were written and published. Among them was the topic of this paper, the first Icelandic textbook in geometry, published in 1889, *Flatamálsfræði* / *Plane Geometry* by the Reverend Halldór Briem, teacher at the new lower secondary school in Northern-Iceland.

GEOMETRY TEACHING IN EUROPEAN HISTORY

The study of geometry was collected into a coherent logical system by Euclid in his *Elements* in 300 BC. The main goal of studying classical Euclidian geometry with its logical deductive axiom system has been considered to train logical reasoning. The Euclidian system provided a model for creating various axiom systems in the last half of the 19th century. Axioms were developed for the set of positive integers by G. Frege and G. Peano in the 1870s, and e.g. Dedekind contributed to a precise definition of the idea of a real number in the same period.

There were, however, several flaws in Euclid's system, e.g. an assumption concerning continuity, not explicitly mentioned. D. Hilbert published his *Grundlagen der Geometrie* in 1899, where he defined five sets of axioms, a complete set, from which Euclidian geometry could be derived. Hilbert's set of axioms contains two which concern the basic idea of continuity, where the tacit assumption of Euclid is made explicit (Katz, 1993: 718–721).

THEORIES ON GEOMETRY LEARNING

According to the theory of Pierre and Dina van Hiele, developed in the late 1950s, pupils progress through levels of thought in geometry. Their model provides a framework for understanding geometric thinking (Clements, 2003: 152–154). The theory is based on several assumptions; that learning is a discontinuous process characterized by qualitatively different levels of thinking, that the levels are sequential, invariant, and hierarchical, not dependent of age, that concepts, implicitly understood at one level, become explicitly understood at the next level, and that each level has its own language and way of thinking.

In the van Hiele model, *Level 1* is the visual level in which pupils can recognize shapes as wholes and cannot form mental images of them. At *level 2*, the descriptive, analytic level, pupils recognize and characterize shapes by their properties. At *level 3*, the abstract/relational level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand, and sometimes even provide logical arguments in the geometric domain, whereas at *level 4*, students can establish theorems within an axiomatic system.

According to Clements (2003), research generally supports that the van Hiele levels are useful in describing pupil's geometric concept development, even if the levels are too broad for some tastes. The van Hiele levels may e.g. not be discrete. Pupils appear to show signs of thinking at more than one level in the same or different tasks in different contexts. They possess and develop competences and knowledge at several levels simultaneously, although one level of thinking may predominate.

GEOMETRY IN EUROPEAN SCHOOLS

The Euclidian axiomatic deductive presentation of geometry was a norm for the subject in early modern age secondary schools. When people began to talk about geometry teaching, based on observation and experiments, by the end of the 18th century in Denmark the idea was hard to fight for (Hansen, 2002: 106).

As a germ to a new era, the proponent of the Enlightenment movement, Rousseau, wrote in his *Emile* in 1762: "I have said that geometry is not within the reach of children. But it is our fault. We are not aware that their method is not ours, and that what becomes for us the art of reasoning, for them ought to be only the art of seeing" (Rousseau, 1979:145). There is a consonance in this quote to the van Hiele theory; the children are still at *level 1*, the visual level.

During the 19th century and the early 20th century the prevailing view of geometry instruction and general education in England was challenged (Prytz, 2007: p. 41–42). Mathematicians like Bertrand Russell resumed the critique regarding tacit assumptions and lack of rigor in Euclid's *Elements*. Educators argued that geometry could be made more palatable to pupils, and others demanded that mathematics instruction should be adapted to practical matters. The last group of critics was led by a mathematics teacher, John Perry, at a technical college where the final examinations had to adhere to the standards of pure mathematics. His efforts were successful and led to changes of regulations in the early 20th century. During the 20th century both practical geometry and the experimental approach were indeed picked up at secondary schools and colleges in England.

As early as 1802 the German philosopher and pedagogue Herbart (1776–1841) argued that imaginary skills are important in connection to geometry instruction. The textbook writers Treutlein (1845-1912) in Germany and Godfrey (1876-1924) in England were influenced by him. Both of them underscored the importance of developing intuitive thinking in connection to mathematics instruction (Prytz, 2007: p. 43–44). At the beginning of the 20th century the German educational system went through important changes, where e.g. the increasing importance of technology undermined the position of pure mathematics. One of the leading actors was the German mathematician Felix Klein (1849–1925) (Prytz, 2007: p. 40).

Thus experimental and intuitive approaches to geometry instruction in secondary schools were discussed in Germany and England by the turn of the 20th century, where influential opinion makers were Perry and Klein. In both these countries, official reports occurred that stressed the importance of such teaching methods and they were included in the first geometry courses at the secondary schools (Prytz, 2007: p. 43).

THE POLITICS OF MATHEMATICS EDUCATION IN ICELAND

In the first half of the 19th century, in 1822–1862, the secondary Reykjavík School, the only school of its kind in Iceland, was served by mathematician B. Gunnlaugsson. Gunnlaugsson had won a gold medal at the University of Copenhagen and made the great feat on his own to measure Iceland geodetically in twelve summers to create the outlines of the country's modern map. During Gunnlaugsson's period, classical geometry teaching was developed at the school according to new requirements of the University of Copenhagen of 1818. Gunnlaugsson had to use Danish textbooks but in order to enhance the pupils' motivation he gave them geodesy problems (Bjarnadóttir, 2006: 90–93; National archives, Bps. C. VII, 3a).

Secondary schools in Denmark were split into a language-history stream and a mathematics-science stream in 1871. The Reykjavík School adhered to the same law, but with own regulations. It was too small to be divided into two streams so after some lobbyism and compromises the school became a language stream in 1877 and mathematics was only taught for four years out of its six-year programme (Bjarnadóttir, 2006: 112–118). This decision caused some dispute and a conflict for several years. University student F. Jónsson, later professor in philology at the University of Copenhagen, wrote in 1883, criticizing the school and its regulations:

> ... to teach mathematics without practical exercises ... is ... as useless as it can possibly be, ... *the worst has been the lack of written exercises*; … all deeper understanding has been missing, all practical use has been excluded ... the new regulations have 1) snapped trigonometry away, 2) prescribed that mathematics is only to be taught during the 4 first years (previously all) and thereby dropped for the graduation examination, and 3) geometry shall commence already in the lowest class;
>
> these three items are as I conceive them equally many blunders; …to skip the trigonometry is to skip what is the most useful and interesting in the whole bulk of mathematics ... that the [geometry] study is to commence in the first grade; in order to grasp it, more understanding, more independent thought is needed than those in the first grade master; [I] tutored two boys in geometry and both of them were not dumb and not merely children, and for both of them it was very difficult to understand even the simplest items; but the reason was that they did neither have the education nor the maturity of thought needed to study such things, which is very natural (Jónsson, 1883: 115–116).[1]

The pupils of the Reykjavík School were sons of farmers, priests and other officials. The priests also made their living from farming as did county magistrates so the majority of the pupils came from farming communities. There were no primary schools in rural areas. The novices came to school prepared by priests in Latin, Danish and basic arithmetic. Presumably most of them had never met geometrical concepts. For example, land properties were not measured in square units, but were from medieval times valued according to how much livestock they could carry.

Considering the van Hiele theory, one may understand that the pupils did not possess 'the maturity of thought' needed to study deductive geometry as presented in the Danish author Jul. Petersen's system of textbooks, written in the period 1863–1878 and used at the Reykjavík School at the time Jónsson is referring to. The pupils were expected to jump to *level 3* of geometric thinking without any preparatory training at lower levels. In Petersen's necrology it said:

> First around the turn of the century people began to realize that the advantages of these textbooks were more obvious for the teachers than for the pupils ... the great conciseness and the left-out steps in thinking did not quite suit children (Hansen, 2002: p. 51).[2]

Petersen's textbook on introduction to geometry, remained as an introductory course at the school for close to hundred years, to be discarded in the late 1960s (Bjarnadóttir, 2006: 320), and was to disturb the life of many a young pupil.

GEOMETRY BY HALLDÓR BRIEM

The Reverend Halldór Briem (1852–1919), published his *Flatamálsfræði* / *Plane Geometry* in 1889. Briem was admitted to Reykjavík School in 1865 to graduate in 1871. He enjoyed there the controversial mathematics teaching described by Jónsson above. Briem was educated as priest in Iceland, but stayed during 1876–1881 in the Icelandic community in Manitoba and Winnipeg in Canada where he was editor of an Icelandic journal. He may have become acquainted there with school mathematics, but record of that is not available. H. Briem wrote textbooks on geometry, English, Nordic mythology, Icelandic grammar and Icelandic history, in addition to theatre plays and various translations into Icelandic, among them of the story of Robin Hood.

In the foreword to the *Plane Geometry*, Briem declared his policy:

> ... no textbook in geometry in Icelandic has been available. I have therefore had to make use of foreign textbooks ... Other schools for the public in this country have not been in better situation in this respect and this shortage is the more severe, as knowledge of mensuration is completely indispensible in various daily tasks of farmers, carpenters and others, besides that it is an important item in general education ...
>
> In composing it, my goal has mainly concerned what is the most important in general industrial activity and therefore I have emphasized the main items concerning that as much as possible, and omitted other items that are less important to the production. The arrangement of the content is therefore different from what is customary in this kind of textbooks, where every sentence is supported by scientific proofs, but according to my policy that did not apply here.
>
> ... [Reykjavík School] teacher Björn Jónsson has read the manuscript of the book, and offered me many good hints ... (Briem, 1889: iii-iv).[3]

H. Briem's brother, the Reverend E. Briem was also a textbook writer. His *Arithmetic* (1869) was a dominating textbook for adolescents, also at the Reykjavík School, in 1869–1910s. It is very unlikely that the brothers were involved in didactic discussions known in Europe about mathematics as a discipline exclusively to train the mind. The brothers declared as their first aim to meet the immediate needs of young people for practical knowledge. One might even conjecture that the authors thought that bothering about proving self-evident facts was an intellectual luxury (or adversity) that educationally-deprived youth were not to be disturbed with.

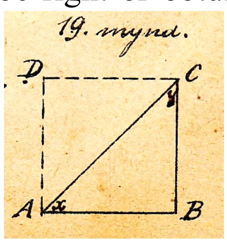
The introduction of H. Briem's *Plane Geometry* is devoted to basic assumptions, such as of a space, a body, a plane or surface, a line and a point, in this order. The body is not composed of planes, the author states, and the plane not of lines as the planes have no thickness. The line has no width and it is not composed of points. However if one thinks of a point moving from one spot to another its track is a line. If a line moves in a direction perpendicular to itself, its track will be a plane and if a plane moves in a direction perpendicular to itself, its track will be a solid.

H. Briem seems to have thought of points as discrete objects and a line as a continuous track, which he could not think of as made up of points. Briem had little opportunity to become acquainted with modern ideas of real analysis or the works of Dedekind or Cantor in the 1870s, and the work of Hilbert on new sets of axioms for Euclidian geometry, where the ambiguity about continuity was amended, had not yet appeared. But a priest teaching mathematics to adolescents on the outskirts of Europe felt a need to philosophize on his own about the nature of lines and planes and their relations to points.

Briem continued with definitions; of parallel lines, an angle, of plane figures, such as triangles, various quadrilaterals, polygons the circle and the ellipse, various quadrilaterals and finally of similarity and congruence. The names of the shapes are in Icelandic with Latin in parentheses. As this was the first book on geometry ever written in Icelandic, remembering it all must have been a difficult task. A score of exercises follow the definitions. Attached to the exercises are answers to them and explanations. This was necessary as lower secondary schools were scarce and the textbook was to serve for home studies as well.

In connection to the definition of a triangle, its attributes are also investigated. It says:

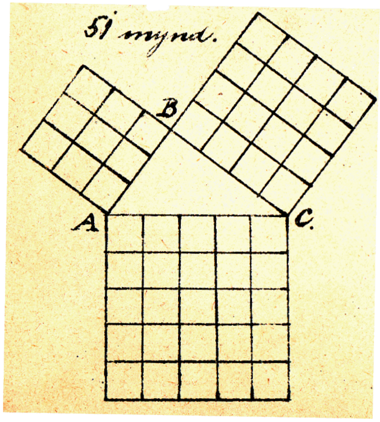
> All the angles in a triangle are 180° in total. In the triangle ABC (diagram 19) CB is perpendicular to AB, therefore the angle B = 1R [R a right angle], furthermore CB is equal to AB; by drawing the triangle ADC equally large and similar to the triangle ABC [congruence had not yet been defined], one may see that x and y each are the half of a right angle, therefore the sum of the angles in the triangle is 2R. The same concerns all triangles, as the larger or smaller one of the angles is, the others (one of them or both) become smaller or larger. In a triangle therefore only one angle can be right or obtuse (Briem, 1889: 14).[4]



In this text a diagram is referred to, but because of a high printing cost all diagrams are printed together as an attachment at the back of the book. Clearly the author appeals to the intuition of the reader to see that the angles x and CAD are complementary, as well as y and ACD. Furthermore, the triangle ABC is a special case of an isosceles right triangle, and the reader is invited to take its attributes as universal. The author had presented parallel lines and their angles to a transversal line and so was able to present the regular proof of the sum of angles in a triangle but obviously he preferred to do it this way.

The common reader, the future farmer or carpenter, may not have been expected to need more 'scientific' proofs, the fact that the sum of the angles in the triangle ABC is two right angles, is more or less obvious from the diagram, but more credulousness is needed for believing that it applies to all triangles. Schools, through the centuries, have expected their pupils to believe what is stated in textbooks. This is not much different from any other point of view than that mathematics studies are expected to foster critical thinking among their students.

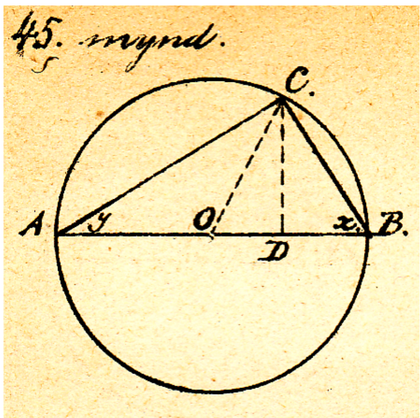
In continuation, a square root is introduced as are common measuring units, which were quite complicated before the introduction of the metric system in 1907. The following chapter concerns areas of parallelograms, squares, rhombi and triangles with plausible explanation aided by the diagrams at the back of the book. The areas of a trapezoid and polygons are deduced from the area of a triangle. Heron's rule is introduced without a proof or explanation, as is the Pythagorean Theorem whose proof is stated to be too difficult for the readers. A diagram of the 3 – 4 – 5 triangle (diagram 51) is presented as an illustration of the rule.



In a circle the perimeter is stated to be $3\frac{16}{113}$ times the radius, while later this and other values for π are said to be approximations to the true value which may be reached as close to as desired. The circle is thought to be composed of many small triangles, whose top-angles meet at the centre of the circle, from which the area of a circle was deduced. This continues with areas of sectors and annuli and finally of an ellipse.

A chapter is devoted to proportions, which probably was difficult as the pupils may not have had much experience in solving equations. When coming to proportions in the right triangle, the author reveals the algebraic proof of the Pythagorean Theorem.

In the final chapter, the author introduced constructions; to bisect a segment, to divide a segment into any number of segments, to construct a right angle, to double the area of a square and a circle and to transform a rectangle to a square with the same area. This is illustrated in diagram 45 where the dimensions of the rectangle are AD and DB and the side of the square is CD. This is a consequence of proportions in the right triangle already introduced, and the author



refers to it through diagrams. Earlier, the necessary prerequisite, that a periphery angle is half the centre angle of the same arc, had been illustrated for a right periphery angle, sufficient for this construction.

All things considered, the text, after the initial introduction of concepts, is readable, although concise, with sensible explanations of most of the formulas with the aid of diagrams, which regrettably could not be attached to the text in concern. The exercises were mainly computations of sizes of angles, lengths of sides in right triangles and various area computations, but no constructions. One may suggest that the level of the book was closer to van Hiele *level 2* than e.g. Petersen's textbook, but was certainly not *level 1*.

However, even if one can claim that Briem's geometry was based on observations of his diagrams, it can hardly be maintained that they concerned the pupils' real world. The problems seldom had content, and if so they were synthetic in the sense that they asked to find areas that few would want to know. It was not customary to compute the area of land except to estimate the time needed to mow it, and few had reasons to find the area of an ellipse-shaped dining table. The author was indeed faithful to the Euclidian content but was unafraid to simplify proofs and appeal to intuition.

The author of *Plane Geometry* taught mathematics, Danish, singing and physical education in the state-run lower secondary school in Northern-Iceland. The *Plane Geometry* was used in that school and possibly in some other schools, but not at the Reykjavík School which adhered to regulations for Danish Latin schools. However, Briem's second geometry textbook on volumes (Briem, 1892), which was not as sensitive to rigor, was used there for some number of years.

In 1904 a learned mathematician, Dr. Ó. Daníelsson graduated from Copenhagen University and came up to Iceland to teach. He completed his doctoral degree in 1909 with geometry as his special field. Until his time there was no mathematician to dispute geometry instruction with. Dr. Daníelsson tried to use Briem's *Plane Geometry* in teacher training for one year, but gave up, presumable due to lack of rigor. He turned to foreign textbooks until he published his own, where he for example used the definitions of parallel lines and their angles to a transversal line to prove that the sum of the angles in a triangle is 180°. He also proved the theorem of Pythagoras with the aid of geometric figures (Daníelsson, 1914).

DISCUSSION

Many pedagogues emphasize that learning is dependent on a cultural environment (see e.g. D'Ambrosio, 2001). It is notable that through the history of education in Iceland, trigonometry and geodesy stand out as being considered interesting and useful subjects, while no trace is found of rigid Euclidian geometry for any other purpose than fulfilling the requirements of the University of Copenhagen.

H. Briem belonged to a generation of intellectuals who were much aware of the low status of education in Iceland and who participated in the struggle for independence in order to be able to form own educational policy. Briem was one of two teachers who were appointed to a new lower secondary school, to whom people had great expectations that it would raise the level of education of the general public. The school was not restricted by any regulations on the mathematics content so Briem had freedom to form the mathematics instruction as he thought suitable. His efforts to avoid 'scientific proofs' reminds of the efforts of Perry in England to release the technical schools from the standards of pure mathematics.

Briem's textbook may be considered as a reaction to geometry instruction as it was performed in the Reykjavík School in the 1870s, without consideration to the young pupils' level of thinking and without any reference to their environment or to the Icelandic culture. It is though questionable to which degree Briem succeeded in connecting the content to the environment and Icelandic reality.

One can hardly claim either that Briem succeeded entirely in meeting the pupils' level of geometric thinking, but he did avoid bothering them with proving what they might have thought 'obvious facts'. His collection of exercises did not contain any pure deduction, but consisted of fairly approachable numerical exercises.

These were times of rapid changes away from a stagnant agricultural society. Craftsmen were a rising class in the 1890s and the textbook was intended to introduce them to basic facts of geometry, useful in their trade. It must have been useful, even if it also contained some irrelevant topics, when taken into account that no other text on the subject was available in their own language. Briem made a great effort to transform concepts from foreign languages into Icelandic, which had no tradition of geometry.

Briem's textbook was indeed an ambitious textbook for its time and no textbook, written in Icelandic, intended for the non-college-bound general public and reaching that level of complexity, has been published since in Iceland.

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[1] ... að kenna stærðfræði án verklegra æfinga ... er ... svo gagnslaust sem frekast má verða, ... það sem vestu hefur gegnt er skortur á skriflegum æfingum ... alla dýpri eigna skilning hefur vantað ... nýja reglugerðin hefur 1) kippt burtu þríhyrningafræði 2) lagt það fyrir að stærðafræði sje aðeins kennd 4 fyrstu árin (áður öll) og þar með slept til burtfararprófs og 3) lagt það til að rúmmálsfræði skuli kennd strax í neðsta bekk;

þetta þrennt er nú að minni hyggju jafnmörg axarsköft; ... að sleppa þríhyrningafræðinni er að sleppa sem því einna nytsamlegast er og skemmtilegast í allri stærðafræðinni ... að námið skyldi byrja í 1. bekk; til þess að nema hana þarf meiri skilning, meiri sjálfstæða hugsun, heldur en þeir hafa almennt, sem eru í neðsta bekk; [eg veitti] tilsögn tveimur piltum í rúmmálsfræði og vóru þeir báðir óheimskir og ekki hrein börn að aldri, og áttu þeir mjög erfitt með að skilja hið allra einfaldasta; en það kom til af því að þeir höfðu eigi þá menntun nje hugsanaþroska, sem þarf til að læra slíkt, og er það fulleðlilegt.

[2] Først henimod aarhundredeskiftet begyndte man at faa Øjet op for at det fortrinlige ved disse Lærebøger var mere indlysende for Lærerne end for Eleverne ... den store Kortfattethed og de udeladte mellemled i Tankegangen ikke egnede sig rigtigt for børn.

[3] ... ekki hefur verið til á íslensku nein kennslubók í rúmmálsfræði. Jeg hef því orðið að notast við útlendar kennslubækur ... Aðrir alþýðuskólar hjer á landi eru ekki betur staddir í þessu tilliti, og er þessi skortur því tilfinnanlegri, sem þekking á mælingum er alveg ómissandi í ýmsum daglegum störfum fyrir bændur, smiði og fleiri, auk þess sem hún er mjög mikilsvert atriði almennrar menntunar. ...

Við samningu hennar hef ég einkum haft fyrir augum, hvað þýðingarmest væri í almennu starfslífi, og dregið því mest fram þau aðalatriði, sem þar að lúta, en sleppt hinu, sem hefur minni þyðingu í starfslífi manna. Niðurskipun efnisins er því nokkuð á aðra leið, en vant er að hafa í þess konar kennslubókum, þar sem hver setning er rakin með vísindalegum sönnunum, en samkvæmt stefnu minni átti það ekki við hjer.

... kennari Björn Jensson hefur lesið yfir handritið af bókinni, og gefið mjer ýmsar góðar bendingar ...

[4] Öll hornin í þríhyrningi eru samtals 180°. Í þríhyrningnum ABC (19. mynd) er CB lóðrjett á AB, þess vegna er hornið B = 1R, ennfremur er CB jafnstór AB; með því að draga þríhyrninginn ADC jafnstórarn og eins lagaðan og þríhyrninginn ABC, má sjá, að x og y eru hvort um sig helmingur af rjettu horni, fyrir því eru öll hornin í þríhyrningnum ABC samtals 2R. Sama á sjer stað í öllum þríhyrningum, því eftir því sem eitt horn er stærra eða minna, verða hin, (annað eða bæði) minni eða stærri. Í þríhyrningi geta því einungis eitt horn verið rjett eða sljóvt.

INTRODUCING THE NORMAL DISTRIBUTION BY FOLLOWING A TEACHING APPROACH INSPIRED BY HISTORY: AN EXAMPLE FOR CLASSROOM IMPLEMENTATION IN ENGINEERING EDUCATION

Mónica 13

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Probability and random variables turn out to be an obstacle in the teaching-learning process, partly due to the conceptual difficulties inherent in the topic. To help students to get over this drawback, a unit on "Probability and Random Variables" was designed following the guidelines of the European Higher Education Area and subsequently put into practice at an engineering school. This paper focuses on the design, implementation and assessment of a specific activity of this unit concerning the introduction of the normal probability curve from a teaching-learning approach inspired by history. To this purpose a historical module on the normal curve elaborated by Katz and Michalowicz (2005) was adapted to develop different aspects of the topic.

Keywords: probability, normal distribution, European Higher Education Area, teaching-learning materials on history of mathematics.

INTRODUCTION

Teaching probability and random variables turn out to be essential for the introducing of statistical inference in any undergraduate course in basic statistics. Statistics is one of the compulsory undergraduate subjects included in the syllabus of any engineering school. This subject, as developed at the School of Agricultural Engineering of Barcelona (ESAB) of the Technical University of Catalonia (Spain), primarily encompasses Data Analysis and Basic Statistical Inference. We believe that the very nature of the subject calls for special consideration in the teaching of the subject, especially with regard to the new European Higher Education Area (EHEA). Besides, the essentially biological profile of the ESAB seems to weaken interest in the mathematical domains.

From our experience in teaching statistics at different engineering schools, we are well aware that probability and random variables represent a rather overwhelming obstacle for students, due to the conceptual difficulties inherent in the topic. To help students get over this drawback, a unit on "Probability and Random Variables" was designed following the guidelines of the EHEA. Subsequently, this unit was put into practice at the ESAB. Throughout the module, the teaching-learning process was assessed using several evaluation techniques so as to analyse the learning outcome achieved (Blanco & Ginovart, 2008). This paper focuses on the design, implementation and assessment of a specific activity of this unit concerning the introduction of the normal probability curve and some related aspects from a historical dimension.

Mathematical and statistical topics have been traditionally taught in a deductively oriented manner, presented as a cumulative set of "polished" products. Through a collection of axioms, theorems and proofs, the student is asked to become acquainted with and competent in handling the symbols and the logical syntax of theories, logical clarity being sufficient for the understanding of the subject. As a result, the traditional teaching of mathematics tends to overlook the mistakes made, the doubts and misconceptions raised when doing mathematics, detaching problems from their context of origin. However, since the construction of meaning is only fulfilled by linking old and new knowledge, the learning of mathematics, in general, and statistics, in particular, lies in the understanding of the motivations for problems and questions. In this respect, integrating the history of mathematics in education represents a means to reflect on the immediate needs of society from which the mathematical problems emerged, providing insights into the process of constructing mathematics (Tzanakis & Arcavi, 2000; Swetz et al., 1995).

How to introduce a historical dimension in our unit on probability and random variables turned out to be a challenge to our "standard" teaching activity, all the more so because first we had to determine which role history would play in the unit. Of the three different ways suggested by Tzanakis & Arcavi (2000) to integrate history in the learning of mathematics, the one that seemed to serve our purpose best was to follow a teaching-learning approach inspired by history. In the context of this paper history was integrated implicitly, since the main aim was to understand mathematics (statistics, in particular) in its modern form, bearing in mind, throughout the teaching process, those "concepts, methods and notations that appear later than the topic under consideration" (Tzanakis & Arcavi, 2000, p. 210). Accordingly, after having selected a historical module on the normal curve elaborated by Katz and Michalowicz (2005, pp. 40-57), we adapted it to develop different aspects of the topic. The aims of the activity were to:

Aim 1.- Show motivation for the topic.

Aim 2.- Show interrelation between mathematical domains, on the one hand, and mathematical and non-mathematical domains, on the other.

Aim 3.- Compare modern "polished" results with earlier results.

Aim 4.- Produce a source of problems not artificially designed for the purpose.

Aim 5.- Develop "personal" skills in a broader educational sense.

These aims are explicitly connected with the ones described by Tzanakis & Arcavi (2000, §§7.2. (a) and 7.2. (c1), pp. 204-206).

THE NORMAL DISTRIBUTION: AN INTRODUCTION INSPIRED BY HISTORY

Right at the beginning of the course our students are informed about the specified learning outcomes, classified according to Bloom's taxonomy (Bloom, 1956) into: Knowledge, Comprehension and Application. The learning outcomes regarding the normal distribution have been articulated as follows:

Table 1. Learning outcomes regarding the normal distribution.

After attending the course the student will be able to:

a) Define and recognize the normal (or Gaussian) distribution, as well as the standard normal distribution.	[Knowledge]
b) Convert an arbitrary normal distribution to a standard normal distribution.	[Comprehension]
c) Calculate probabilities of events when a normal distribution is involved, using the table of the standard normal distribution.	[Comprehension]
d) Describe the empirical rule 68-95-99.7.	[Comprehension]
e) Apply the rule 68-95-99.7 to assess whether a data set is normally (or approximately normally) distributed.	[Application]
f) Estimate the approximation of the normal distribution to the binomial distribution.	[Application]

To adapt the historical module it was first necessary to frame the activity within well-defined boundaries (Katz & Michalowicz, 2005). Therefore, we started selecting and later reflecting on some questions suggested by Pengelley (2002) for assessing historical material: (a) What is the purpose of studying the material? (b) How does it fit in with the curriculum? (c) Are there appropriate exercises, with an appropriate difficulty level and well chosen to demonstrate concepts? (d) Will it motivate students? (e) Will it help with something students have trouble with? Since the activity described in this paper was directed towards the learning outcomes mentioned above (see Table 1), question (b) was explicitly involved.

To show the original motivation for the topic of the normal distribution, the activity emphasized interrelation between statistics and health and social sciences, hence covering Aims 1, 2 and 4. Although the topic had already been introduced in the classroom, the teaching-learning process was able to benefit from the study of non-artificially designed problems. From Katz's module we elaborated the material for the activity combining information about the historical development of the normal curve with some "appropriate" questions. There were no accompanying answer sheets as the activity was designed to be worked out in a two-hour computer lab session, individually or in pairs. Most of the students worked individually, whereas only few computers were shared by two students working together. The teacher acted as a consultant during the session. Students managed the time given over to every section of the activity themselves, according to their individual needs and skills. If they could not accomplish their work in the computer lab, they had the possibility to do it as homework. It is worth pointing out that the questions were chosen not only to assess understanding of the information provided, but also to bring out the connection with other mathematical domains. Hence, students were asked to prove expressions and formulae, to use a spreadsheet to carry out elementary probability calculations and to represent data, and to investigate supplementary aspects regarding the contents of the activity. All these aspects were planned in order to cover Aims 3 and 5.

In connection with question (a) stated above, this activity attempts to introduce the normal probability distribution in its original context, and to help students to get acquainted with basic calculations involving the normal curve. The first section of the activity shows how De Moivre (1667-1754) obtained his discovery of the empirical rule 68-95-99.7. The second section gathers the discussion on the error curve in which Laplace (1749-1827) and Gauss (1777-1855) were involved. How Quetelet (1796-1874) calculated the table of the normal distribution from the approximation of the normal distribution by the binomial distribution is the target of the third section. To close the activity, the fourth section is centered on the first uses of the normal distribution in the real world, namely: i) analysis of the chest circumference of 5732 Scottish soldiers; ii) analysis of the heights of French conscripts to assess the normality of the distribution, revealing a significant figure of men who illegally avoided recruitment.

We interspersed the text with seven leading questions related to the topics discussed, given at strategically points during the activity, and not on a separate sheet at the end. *Questions 1*, *4*, *6* and *7* were directly inspired by the ones suggested by Katz and Michalowicz (2005) on pages 46, 55, 56 and 57, respectively. The rest were stated by us, to ensure that a particular point was fully understood. The questions were conveniently placed after a specific topic or a related result. The following paragraphs briefly describe each question, drawing attention to the educational aims served by each one.

Question 1: In an experiment in which 100 fair coins are flipped, about how many heads would you expect to see? What is the corresponding standard deviation? Find the limits (lower and upper) for the number of heads we would get 68%, 95% and 99.7% of the times.

This first question deals with direct manipulation of a binomial distribution, followed by a first encounter with the connection between the normal and the binomial distributions. This was intended to help students to “warm up” by stating a link between the activity and a topic they had already learned in the classroom, thus relating to Aim 1.

Questions 2 through *4* are connected with Quetelet's calculation of a symmetric binomial distribution. He considered the experiment of drawing 999 balls from an urn containing a large number of balls, half of which were white, and half black.

> *Question 2*: Prove Quetelet's shortened procedure for the calculation of relative probabilities: $P(X = n+1) = \frac{999-n}{n+1} \cdot P(X = n)$, where $P(X = n)$ represents the probability of drawing *n* black balls from the urn. Setting the value of $P(X = 500)$ to be 1, calculate the relative probabilities $P(X = 501)$ and $P(X = 502)$.

Students had to deduce this recursive formula from the probability function of the binomial distribution. This question was inserted to show the interrelation between mathematical domains, namely, probability and recursive proofs (Aim 2). In this case the interest lies in how to evaluate mathematical arguments and proofs, and to select and use diverse types of reasoning and methods of proof as appropriate (Ellington, 1998). Given that students often meet difficulties in proving recursive formulae, this exercise seems to be consistent with questions (c) and (e) suggested above.

> *Question 3*: Using an Excel worksheet recalculate column A of Quetelet's table for the values 500 to 579 and graph the corresponding curve.

To get a deeper knowledge of the binomial-normal link, students were here asked to use a spreadsheet, in particular, the spreadsheet program Microsoft Excel. Since the activity was developed in the context of computer practicals, students had computers at their disposal. The computer practicals offer students the possibility to be actively engaged in the learning process, as well as to apply the concepts learnt to the prospective working practice. Since this topic turns out to be a usual source of difficulty, this exercise connects again with question (e). Besides, it helps not only to compare modern results with earlier ones, but also to develop "personal" skills such as how to manipulate a spreadsheet. Therefore, this exercise focuses on Aims 3 and 5.

> *Question 4*: A discrete variable can be approximated by a continuous variable considering the following estimation:
>
> $P(x = k)_{discrete} \approx P(k - 0.5 \leq x \leq k + 0.5)_{continuous}$.
>
> For instance, $P(x = 500)_{binomial} \approx P(499.5 \leq x \leq 500.5)_{normal}$.
>
> Using this information, recalculate the first four values in column A using a modern table of the normal distribution.
>
> It can be assumed that the results of drawing balls out of the urn are normally distributed with mean of the number of black balls equal to 500 and standard deviation equal to $\frac{1}{2}\sqrt{999} \approx 15.8$. Compare these results with Quetelet's binomial table.

Understanding why we do things the way we do, and how mathematical concepts, terms and symbols arose, plays a relevant role in grasping the topic (Ellington, 1998). This question allowed the students to compare a modern table of the normal curve with the earliest table. Thus Aim 3 is again involved in the proposed activity.

Finally, *Questions 5, 6* and *7* concern some real world applications of the normal distribution.

Question 5: Read carefully Quetelet's procedure for determining whether the chest circumferences of the Scottish soldiers were normally distributed. Write down those points you do not understand completely.

Question 6: From the results in the example of the heights of French conscripts, discuss how Quetelet concluded there had been a fraud.

From the reading and through understanding of the example on the chest circumferences (*Question 5*) students were to draw conclusions in the case of the heights of French conscripts (*Question 6*). However, as we will see in the following section, since Quetelet's procedure proved to be difficult to understand, only a few students managed to answer *Question 6* correctly.

Questions 4, *5* and *6* contribute to Aim 3 in that they help to compare historical results with modern "polished" ones. Likewise, Aim 4 could be achieved, since these questions convey the idea that probabilistic tools represent a means to solve real-world problems, rather than just artificial designed exercises, framed in a theoretical context. By and large, this set of questions also fosters the practice of reading comprehension skills (Aim 5).

Question 7: On the Internet, browse for the information on Galton's machine. What was the relationship between the inventor Francis Galton (1822-1911) and Charles Darwin (1809-1882)?

The intend of this last question was to help develop some "personal" skills, in a broader educational sense, such as reading, summarizing, writing and documenting (Aim 5). Additionally, it was interesting to point out the interrelation between mathematical and non-mathematical domains, namely, between statistics and the theory of evolution put forward by Darwin (Aim 2). A fundamental part of this question involves the writing component and documenting. The incorporation of a writing component in statistics courses has been encouraged in recent years by Radke-Sharpe (1991) and Garfield (1994). Writing helps students to think about the assumptions behind statistical, graphical or instrumental procedures, to formulate these assumptions verbally, and to critically examine the suitability of a particular procedure based on its assumptions. The inclusion of documenting (i.e. browsing the Internet) facilitates student reading, understanding and summarizing from different sources. In short, reading, writing and documenting are tools that will serve students well in their future scientific or academic writing. Encouraging students to put concepts such as these into words will strengthen their understanding of those concepts.

ASSESSMENT OF THE TEACHING-LEARNING PROCESS

Among the questions mentioned above for assessing historical material, Pengelley (2002) suggests considering whether it will motivate students (question (d)). Though not the only source of feedback, student ratings provide an excellent guide for designing the teaching-learning process and, in particular, for assessing their motivation. Therefore, at the end of the activity students were asked to rate the activity thus:

(1) Very good, (2) Good, (3) Satisfactory, (4) Poor, and (5) Very poor.

Figure 1 shows the results of this survey. Of the 60 students who took part in the activity, half of them regarded it positively (22 satisfactory, 6 good, 1 very good), whereas the other half rated it as poor.

Figure 1. Student ratings on the activity.

Another aspect suggested by Pengelley (2002) for assessing historical material concerned the suitability of the degree of difficulty (question (c)). To determine whether the activity was appropriately difficult, we analysed in detail a random sample of size 20 drawn from the students who had handed in their answers. Every question (except *Question 5*) was marked with either Non-Answered, Poor, Fair or Good. From the graphics of Figure 2 regarding the assessment of the questions, it is clear that *Questions 1* through *4* are most frequently marked as “Good”. Surprisingly, all the students answered *Questions 1* and *2*, whereas the ratio of “Non-Answered” in *Question 6* exceeded the rest of marked ratios. As for *Question 7*, most of the students got “Fair”. This was partly due to the fact that students merely copied the information from the Internet and pasted it on their worksheets, thus showing no interest in summarizing the information in their own words.

Relating to *Question 5*, from the comments given by our students we gathered that the construction of the table proved to be, in general terms, rather cumbersome.

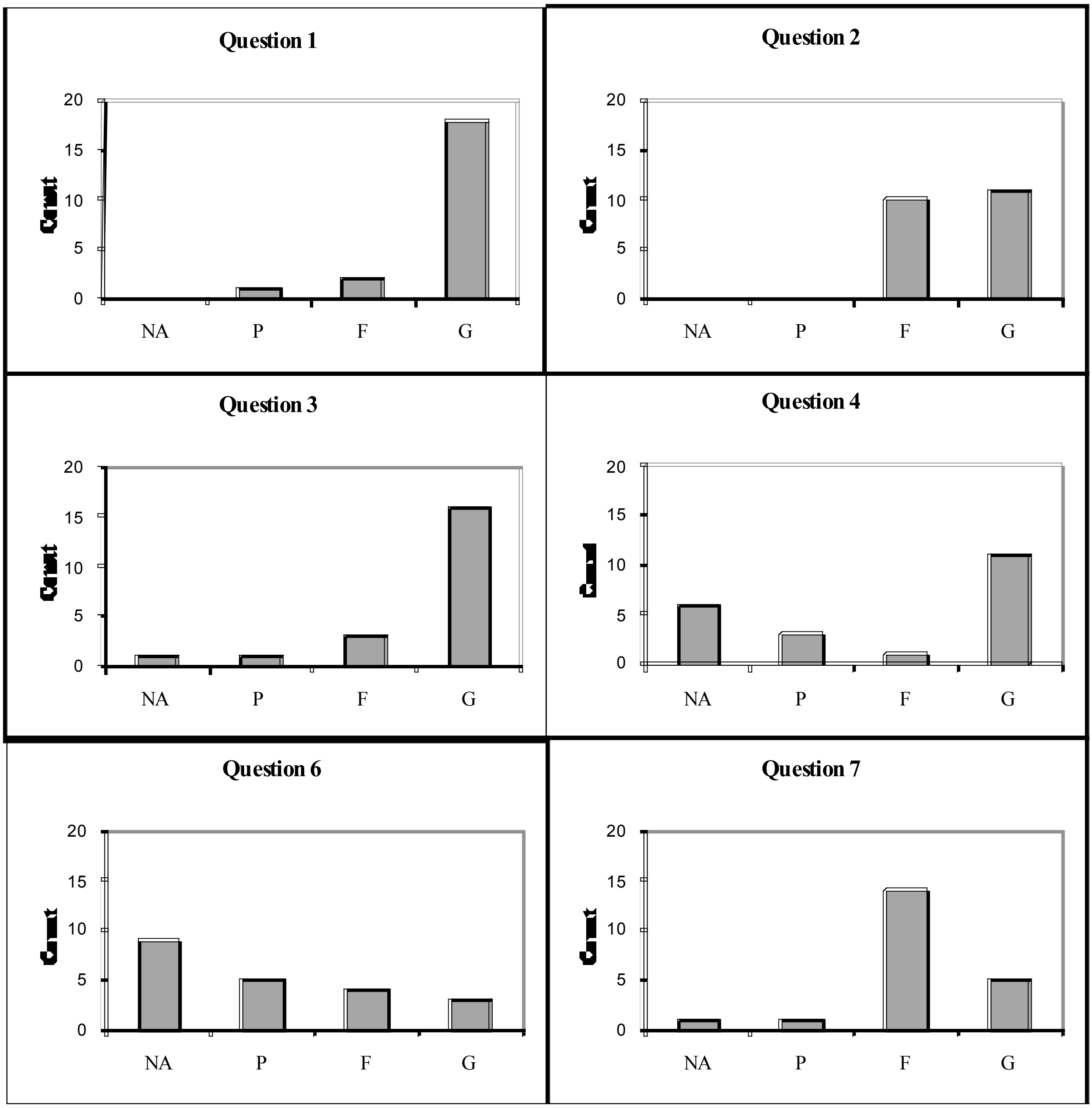

Figure 2. Assessment of the *Questions* of the activity with Non-Answered (NA), Poor (P), Fair (F) or Good (G).

FINAL REMARKS

As Fauvel and Maanen (2000) point out, one should not underestimate the difficult task of the teacher in achieving a proper transmission of historical knowledge into a productive classroom activity for the learner. Given our lack of expertise in the field, in this first experience we were not able to foresee all the possible obstacles in the understanding process. Now we are aware of some difficulties inherent in the material (regarding, for instance, *Questions 5* and *6*). First of all, the mathematical language and form (notation, computational methods, etc) turned out to be rather confusing right from the beginning. In addition, the syllabus and a sense of lack of time made us cram the activity into a two-hour class. Likewise, we had a slight doubt about how useful the topic was for our students. Why not give the opportunity to appreciate the topic in itself, stressing the aesthetics, the intellectual curiosity, or the recreational purposes involved? Finally, we borrowed and adapted part of Katz's historical modules on Statistics, but in keeping with our syllabus, more didactic resource material on this topic should be elaborated for future use.

On the whole, however challenging, the experience proved to be rewarding in the end. Not only did the activity supply a collection of non-artificially designed problems, but it also helped to develop further skills, such as reading, writing and documenting. Above all, it was a means to show the original motivation of the normal curve and hence, to render it more understandable. This experience has shown that probability cannot be regarded as a collection of "polished" products within a deductive structured system, but rather as a system with a peculiar life (expectations, false expectations and false starts), as Guzmán (1993) put it, determined and influenced by external factors and connected with mathematical and non-mathematical domains.

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ARITHMETIC IN PRIMARY SCHOOL IN BRAZIL: END OF THE NINETEENTH CENTURY

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The arithmetic is part of mathematical knowledge based on the idea of the number. The teaching of intuitive calculation in Brazil in primary education level at the end of the nineteenth century and early twentieth century seems to be influenced directly by the "Cartas de Parker". These arithmetic charts based on the ideas of Pestalozzi, Froebel and Herbart were diffused in arithmetic textbooks and educational journals, testimonies of their strong influence in Brazil. This article is based on methodological presuppositions of the Cultural History, of the History of School Disciplines and the studies on the School Culture.

Keys-words: Arithmetic, Intuitive Calculation, Cartas de Parker, Grube's Method, Elementary level.

INTRODUCTION

This article presents one of the partial results of the literature search undertaken within the framework of our thesis of doctorate, still in its phase of development. It aims at carrying out a historical survey of mathematical teaching in Brazilian primary education. We seeks to analyze the part that deals with "counting" in "reading, writing and counting". Furthermore, we want better understand the process of its teaching by seeking answers to questions like: which textbooks were adopted for the teaching arithmetic? Which role held psychology in the evolution of the arithmetic's textbooks for primary education? How were the contents of school arithmetic modified in the textbooks?

By considering the contributions of Cultural History, of the History of School Disciplines and the studies on School Culture, this research privileges the documentary, textbook sources, school files, legislative texts related to teaching as well as the old materials of the daily newspaper (personal records of teachers, books of pupils, tests, school periodics and examination questions) [1].

According to Enfert (2003), contrary to what occurred for research on the history of primary education in French, the history of mathematics teaching at this level did not receive the attention which it deserves. Except for some cases of specialized studies, research, in a general way, mostly treated mathematics teaching at the secondary or tertiary levels. A history of this discipline was thus not treated yet as a whole (arithmetic, geometry, geometrical drawing, algebra, accountancy, etc), nor over its long duration. In the History of School Disciplines, Chervel (1998) defines a particular phenomenon which he calls "vulgata". At each time, the teaching given by the teachers is, *grosso modo*, identical, for the same discipline and the same level. All textbooks, or almost, say more or less the same thing then. The concepts, the adopted terminology, the succession of headings and chapters, the organization of the corpus of knowledge, even the examples or the types of practiced exercises are identical, except for some small variations. These are the variations, which can justify the publication of new textbooks although they present only tiny variations.

The description and the analysis of the “vulgatas” are fundamental tasks for the historian of a School Discipline. If it is not possible to examine the whole of the leading production carefully, it rests with him to determine a corpus sufficiently representative of their various aspects; it is only in this manner, that he can arrive at concrete and conclusive results.

Research in the mathematics teaching in Brazil at primary education level during the end of the XIXth century, particularly related to the textbooks written by representative authors of their community, revealed a reference particular to what is called the “Cartas de Parker”. Indeed, their contents seem a reference and a model adopted by various textbooks published at the beginning of the XXth century, and seem to be constituted in a “vulgata” which influences the teaching of the rudiments of calculation at this level of teaching.

INTUITIVE CALCULATION

According to Buisson (1880), intuitive calculation is a term, which means manner of teaching the first elements of calculation. This methodology borrowed from Germany, was diffused in Russia, in the Netherlands, in Sweden and found a strong adhesion in the United States. This form of teaching was called Grube’s method.

In 1842, Grube published in Berlin the first edition of his *Leitfaden für das Rechnen in der Elementarschule nach den Grunsätzen einer heuristischen Methode* (Guide for calculation in the elementary classes, following the principles of a heuristic method). This “*Essai d'instruction éducative*”, as he called it, after causing warm discussions, was approved by teachers. This book was also in agreement with the new system of weight and measurements and reached in 1873 its 5th edition. Many textbooks, in all languages, reproduced, imitated or applied Grube’s method.

Grube’s method consists in making pupils do themselves, by intuition, the fundamental operations of elementary calculation. Such a method aims at making known the numbers: knowing an object, does not only mean knowing its name, but also apprehend it in all its forms, in all its states, its various relations with other objects; it means being able to compare it with others, to follow the transformations, to write it and measure it, compose it and break it up, at will.

By treating numbers then as unspecified objects to which familiarize pupils, Grube is opposed to the old sequence teaching who consists in learning successively, in first the addition, then the subtraction, finally the multiplication and the division. It devotes the first year of the elementary course to the study of the numbers from 1 to 10; the second year is devoted to the numbers from 10 to 100; the third year being devoted to the numbers from 100 to 1000, and so on, and the fourth and last year, being devoted to the study of fractions.

This methodology does not prepare only the pupil to enter the everyday life and to study the arithmetic, but it offers as advantage over the other methods to meet the conditions necessary to the promotion of mental calculation. The pupils subjected to this method do not become slaves of the numbers, their pencils and their “armed operations”.

Soldan (1878) exposes the six most important points of the Grube’s method of teaching:

a) *Language* - the language is the only means by which the teacher will have access to what the pupil is thinking, because it is not requested any records of the calculations made by them. A complete answer must be required pupil, because it is only by it, that the teacher will be able to evaluate what the pupil has learnt or not.

b) *Questions* - the teacher must avoid asking a great number of questions. The pupils must express themselves as often as possible by themselves.

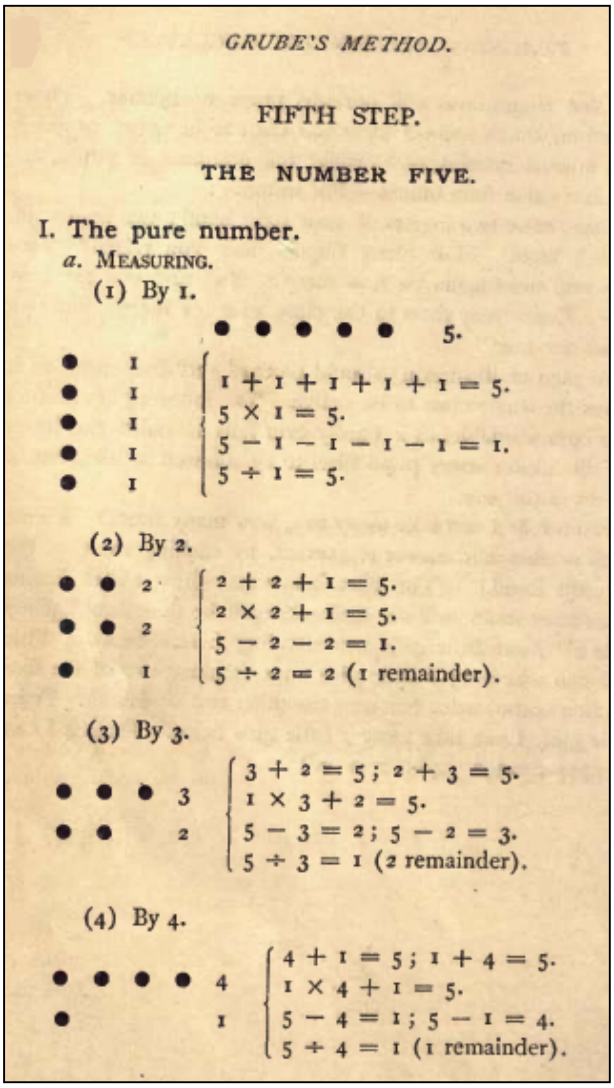
c) *Individual recitation* and *jointly with the class* - In order to bring animation to class, the answers to the questions must alternatively be given individually and in groups, mainly while following the *numerical diagram* [2].

d) *Illustration* - Each process and each example must be illustrated by means of objects that must necessarily be present in the class.

e) *Comparison and measurement* - the contents of each session consist in comparing and measuring each new number with the precedent, by taking account of the relation of difference or quotient, by integrating the four fundamental rules. It must also give sufficient examples associated with this action, not only with what is called the *pure number*[3], but also with the *numbers applied.*

f) *Writing figures* - As the method advances, the pupil must be able to draw the *numerical diagrams*.

The study of Grube’s methodology makes it possible to advance that it had an influence in the publications of Mr. Parker.



GRUBE'S METHOD.

FIFTH STEP.

THE NUMBER FIVE.

I. The pure number.

a. MEASURING.

(1) By 1.

● ● ● ● ● 5.

● 1 | 1 + 1 + 1 + 1 + 1 = 5.
● 1 | 5 × 1 = 5.
● 1 | 5 − 1 − 1 − 1 − 1 = 1.
● 1 | 5 ÷ 1 = 5.
● 1

(2) By 2.

● ● 2 | 2 + 2 + 1 = 5.
● ● 2 | 2 × 2 + 1 = 5.
● 1 | 5 − 2 − 2 = 1.
| 5 ÷ 2 = 2 (1 remainder).

(3) By 3.

● ● ● 3 | 3 + 2 = 5; 2 + 3 = 5.
● ● 2 | 1 × 3 + 2 = 5.
| 5 − 3 = 2; 5 − 2 = 3.
| 5 ÷ 3 = 1 (2 remainder).

(4) By 4.

● ● ● ● 4 | 4 + 1 = 5; 1 + 4 = 5.
● 1 | 1 × 4 + 1 = 5.
| 5 − 4 = 1; 5 − 1 = 4.
| 5 ÷ 4 = 1 (1 remainder).

Fig. 1 – The Grube's Method

INTUITIVE'S METHOD AND THE "CARTAS DE PARKER" (NUMERICAL DIAGRAMS)

Research on the teaching of mathematics in Brazil in primary education at the end of the XIXth century through the sources, revealed a reference particular to Mr. Parker, this eminent American teacher, author of "Cartas de Parker".

According to Montagutelli (2000), Francis Wayland Parker (1837-1902) developed an educational system which was recognized by John Dewey as the "father of progressive education", also inspiring a few years later Granville Stanley Hall. Coming from a family of educators, Parker has been already teacher with sixteen years, and later also served in the army at the time of the Secession War in the United States. At the end of the hostilities, he took the direction of a school in Ohio. In 1872, he made a study trip in Europe: in Germany, he got familiarized with Herbart's pedagogy. It is possible that he took note of the Grube's method then. In 1875, he went back to the United States, where he became superviser of schools in the town of Quincy, Massachusetts. It is during this period that Parker developed the so-called "Quincy System". In an atmosphere removed from the rigid discipline imposed in the majority of the schools of this time, the pupils read newspapers or texts composed by their teachers; starting from what they knew, they approached the new concepts concretely, and then worked in groups; they practiced also drawing and music.

Parker published five books on education: *Talks one Teaching* [4] (New York, 1883); *The Practical Teacher* (1884); *Course in Arithmetic* (1884); *Talks on Pedagogies* (1894) and *How to Teach Geography* (1885).

An important educational journal of the beginning of the XXth century, "Revista de Ensino", created in 1902 by the Association of Public's Teacher of São Paulo, devoted in several numbers, in its section called Teaching Pratice, several articles on the manner of using the "*Cartas de Parker*".

This educational publication, over a number of figures, published about fifty charts, diffusing them in Brazil. These charts concretize the appropriation by Parker of the *numerical diagrams* stated in Grube's method. They represent the manner of treating the teaching of Arithmetic in an intuitive way. Moreover, they are presented in the form of a reference for the development of textbooks of mathematics intended for the first levels.

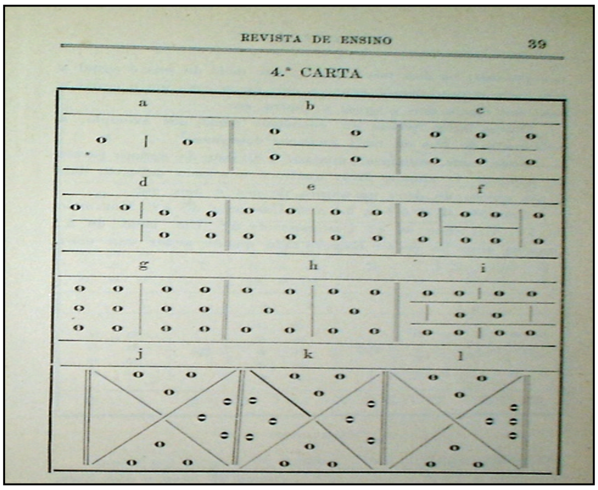


Fig. 2 – 4th Carta de Parker

By a heuristic process, i.e., a procedure, consisting in making the pupil discover what exactly he wants teach him, the teacher questioned the pupil in front of the chart. Example extracted the fourth chart (see Fig. 2): in the items ***h***, ***i*** and ***l*** are drawn representations of the number *ten*. And by the observation, the pupil is brought to give his answers or to make remarks for the formation of this number. Thus, in the letter ***h***, one needs two *five* to form the *ten*; in letter ***l*** we find *three* + *three* + *four* to form the *ten*; in letter ***i***, one needs five times of *two* to form the *ten*. From this way the pupil learned how to compose and break up the number into equal or unequal parts.

The idea of the addition, subtraction, multiplication as of division is concomitantly subjacent with this process.

In Brazil, in addition to the quotations and the articles of "Revista de Ensino" on "Cartas de Parker", an important textbook from the beginning of the XXth century, written by Arnaldo de Oliveira Barreto, *Série Graduada de Matemática Elementar*, published by the Salesians, in São Paulo, in 1912, quotes the name of Parker and the "Cartas de Parker" in the foreword signed by Oscar Thompson, then director of the Normal School (Teacher School). There are also quotations in the presentation of the book and the final comments relating to the conferences pronounced by Parker.

The effective methodology of teaching during this time treated intuitive method which had been adopted in second half of the XIXth century in the European, American and Brazilian schools; it was based on the ideas of Pestalozzi and Fröbel.

For Valdemarin (1998), the intuitive method was influenced directly by the current empirist of philosophy, carried by Francis Bacon and John Locke (XVIIth century) by determining the procedures of teaching based on the *observation*.

This method was presented in the form of a response to the abstract character and little utility of the instruction up to that point of use, by developing new didactic materials and a diversification of the teaching activities. It also brought with it of other innovations as the successive Universal Expositions which were organized for the diffusion of teaching practices, as those which were held in London (1862), in Paris (1867), Vienna (1873) or Philadelphia (1876).

The presence of the intuitive method in teaching of arithmetic reveals a new teaching thought which is opposed to the preceding provisions of teaching where the memorizing of the knowledge was privileged. The "Cartas de Parker" are the elements which made it possible to associate the influence of this intuitive movement of the teaching of arithmetic in Brazil at this time, as attests of it the diffusion of this methodology by means of the disclosure of the educational journals, such as the "*Revista do Ensino*" and of the textbooks like "*Aritmética Escolar*" of Ramon Roca Dordal [5] or "*Contador Infantil*" of Heitor Lacerda [6], among others.

CONCLUSION

According to Chervel (1998), the first task of the historian of the School Disciplines is to study the explicit contents of disciplinary teaching. The study of a "vulgata", configured as that of the "Cartas de Parker" enables us to connect the form and the contents of the teaching of mathematics in the primary education level at the end of the XIXth century - beginning of the XXth century in Brazil, an important element of the writing of the History of Mathematical Education in Brazil.

Moreover, this study makes it possible to clarify the influence of the teaching ideas which circulated at the end of the XIXth century in Europe and which materialize in Brazil in the form of publications of textbooks and articles in educational journals. This direction seems to indicate the influence of intuitive teaching, conceived by their

European authors like a teaching instrument able not only to mitigate the inefficiency of school teaching, but also to reduce differences among the economic development, considering emergent industrial work required instructed people and able to reason quickly and in a creative way.

According to Valdemarin (1998) this inefficiency of school teaching was characterized by the training of pupils who insufficiently controlled the reading and the writing and whose not very satisfactory concepts of calculation, mainly because of the practice to exclusively base the training on the memory, to give the priority to the abstraction, to develop the repetition with the detriment of comprehension and to impose contents without examination and of the discussion.

The explicit proposal of the "Cartas de Parker" seems to be in phase with the one time aspirations which rejects the methods primarily based on the memory and develops the observation like a means of effective training of the training of calculation.

It is through historical studies that we have access the way in which the large teaching thinkers thought the teaching of mathematics and of which echo it had in Brazil.

NOTES

1. This research is subordinated to one of the thematic projects which are developed by the GHEMAT – Grupo de Pesquisa de História da Educação Matemática do Brasil (Group of Search for History of the Mathematical Education of Brazil): "A EDUCAÇÃO MATEMÁTICA NA ESCOLA DE PRIMEIRAS LETRAS, 1850-1950" coordinated by Prof. Dr. Wagner Rodrigues Valente and financed by the FAPESP. Through a financial support obtained from CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico (National Council of Technological and Scientific Development), I have been developed my research of doctorate at INRP/SHE (Institut National Recherche Pédagogique, Service d'Histoire de l'Education – Paris – France) under supervision of Prof. Dr. Alain Chopin (05/2008 to 04/2009).

2. The *numerical diagram* of the Grube's method will be presented later on in this article as the "Cartas de Parker".

3. A *pure number,* also called an *abstracted number*, which is that makes mention only quantity. Four, thirty, twelve are examples of *pure numbers*. Applied to an object, it will be called a *number applied* or *numbers concrete*. Thirty apples, four trees, three meters, are examples of *numbers applied* or *concrete*.

4. This book was translated into portuguese by Arnaldo de Oliveira Barreto in 1909 and edited by Livraria Francisco Alves : "*As Conferências de Parker*"

5. See article Costa, D.A., Valente, W.R. (2007). Análise da Arithmética Escolar de Ramon Roca Dordal. In: *Simpósio Internacional do Livro Didático, 2007, São Paulo. Livro Didático - Educação e memória. São Paulo: Centro de Memória da Educação* – FEUSP, v.1.

6. See *Revista do Ensino*, 1902, p.146.

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HISTORICAL PICTURES FOR ACTING ON THE VIEW OF MATHEMATICS

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The article illustrates the underlying philosophy of an in progress book in which pictures taken from historical books are used to hint some fundamental ideas of the history of mathematics. Both epistemological and disciplinary issues are taken into account. The aim of the book is to let its potential readers know different aspects of mathematics as a science operating inside the socio-cultural context.

Keywords: historical images, original sources, mathematics view.

INTRODUCTION

This paper deals with the problem of the view of mathematics held by students and the means suitable to act on it. In previous works we have studied students' view of mathematics as a socio-cultural process with particular reference to the historical development, see (Demattè & Furinghetti, 1999). Our main conclusion was that this view was very narrow focused and based on common myths on mathematics. To answer the question "How to act on the image of mathematics held by students?" a book has been designed by one of the authors (A. D.) addressed to students of the final years of secondary school (16 years old onward) or readers who are interested in the popularisation of mathematics. The book is based on pictures taken from historical sources. Pictures have been largely used in history for communicating mathematical ideas, see (Mazzolini, 1993), and thus it is not difficult to collect materials for composing such a book. Words accompany pictures in order to create a unitary discourse and to focus on some aspects. Pictures strengthen what the verbal part say, like in a natural history museum where things and words, verbal and non-verbal communication coexist. Knowledge required for using the book in classroom (or elsewhere) is confined to elementary mathematics. As we will see in sections 3 and 4 some chapters are more suitable to develop mathematical topics *stricto sensu*, other are more oriented to raise reflections on historical-epistemological questions.

THE ROLE OF PICTURES

The idea of this book does not come out of the blue. We have already described in (Demattè, 2005; 2006a; 2006b) our work with pictures in the classroom. In particular, in the latter two papers we have discussed how students in front of a historical figure are able to mobilize some kind of narratives and to produce conjectures. This is due to the particular nature of the information provided by figures. Often images show supplemental details, which are not pertinent to the specificity of discourse. Readers can interpret these images in different ways. A discourse follows a logical track (sometimes very rigorous), a picture often permits freedom to the interpreter.

Therefore it is ‘friendly’ i.e. rich in possibility of reflections and personal reasoning. Our claim may be illustrated by some examples taken from the book.

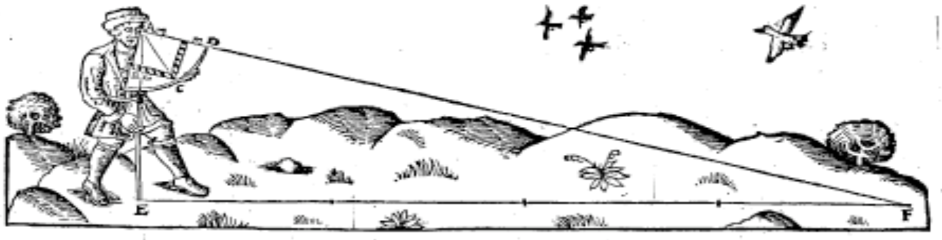

Fig. 1. Oronce Finé, *Protomathesis*, 1532

Pictures like Fig. 1 are aimed at showing how an instrument can be used, but the painter has added many details (hills, grass, trees, birds, elegant dress of the man) which make the scene realistic. The draw of the right-angled triangle and of the instrument (a “quadrant in a fourth part of a circle”) focuses on mathematical aspects.

To reflect on the use of the picture in Fig. 1 in classroom raises the following questions for the researcher: Can students appreciate these kinds of images? Do pictures like Fig. 1 make them want to use the facilities offered by mathematics? Do students see the relationship between the concepts and procedures shown in historical pictures and what they learn in school today? Maybe the answer is no, for each question. In any case the mathematics view suggested by this kind of pictures appears potentially positive in the fact that they address the attention to geometrical details and, in the same time, stimulate guessing the finalities of the action illustrated in the picture. A scene like the one in Fig. 1 suggests a simple story, a narration with a precise structure (some events happen before, some after, a goal of the action – including the implicit use of mathematics - is noticeable). (Demattè, 2006a; 2006b) report on an experiment where students were asked to write how they interpret Fig. 2.

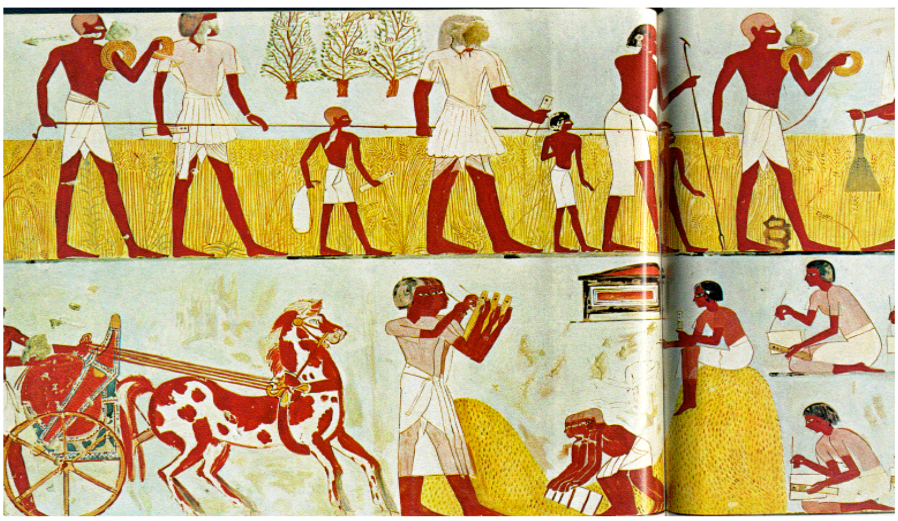

Fig. 2. A mural painted at Abd-el-Qurna, Egypt, around 1400 B.C

Some protocols show that they followed the pattern of a narrative. Because of the need to complete the story, students formulated also conjectures (e.g. the kings' servants on the cart have the task of rewriting the data and, as the student write, "the aim of giving an account of them to the king").

Students are rather naturally brought to formulate conjectures, which are coherent with context and with elements present in the scene, if they have adequate knowledge. To interpret mathematical aspects in the previous image from Finé's *Protomathesis* or in the following Fig. 3 the concept of similarity among triangles is required. But many other aspects require more knowledge: e.g. Why the square? Which is the purpose of the action of the man in the picture? etc.

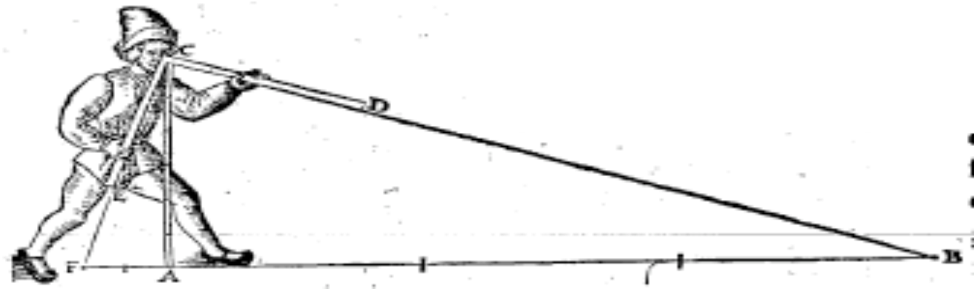

Fig. 3. Oronce Finé, *Protomathesis,* 1532

PICTURES AND MATHEMATICAL TOPICS

In the book the focus is on some grounding mathematical ideas that may be elaborated through the history of mathematics. These ideas regard the main chapters of mathematics (numeration, algebra, probability, etc., see Appendix). Some ideas are inherent to procedures and concepts: images suggest first of all the *incipit* of mathematical reasoning and its global structure. For example, the reader may reflect on the different ways of approaching the same theorem by considering the Chinese theorem of Pythagoras (Fig. 4) and what is done using Cartesian graphs.

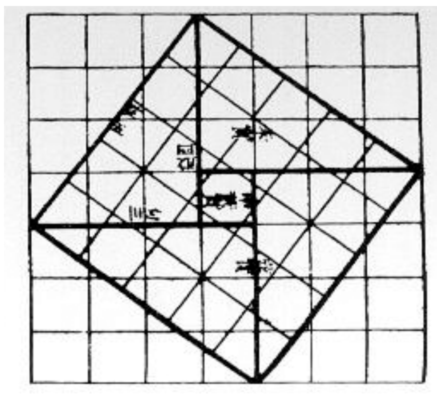

Fig. 4. 'Pythagorean' theorem from *Chou Pei Suan Ching*, about 500-200 b.C.

Moreover pictures, suggest at a glance some metacognitive information e.g. the level of complexity and the need of a detailed mathematical reasoning, as exemplified by the Leibnizian graphs shown in Fig. 5 from *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus* (A new method for maxima and minima as well as tangents, which is impeded neither by fractional nor by irrational quantities, and a remarkable type of calculus for this), see (Dupont & Roero, 1991).

Fig. 5. Gottfried Wilhelm Leibniz, *Nova Methodus* …, 1684

17. PICTURES AND HISTORICAL-EPISTEMOLOGICAL IDEAS

Some chapters address historical and socio-cultural aspects such as: reckoning and measuring as answers to problems of human activities. The students may perceive the hypothetical-deductive structure of mathematics as a model for other branches of the human knowledge such as philosophy and economy, or for every day life. Through these chapters some myths about mathematics may be discussed: the development of mathematics seen as a linear progress from ancient to contemporary times, euro centrism, independence from external factors.

In our previous papers, see (Demattè & Furinghetti, 1999; Furinghetti, 2007) we discussed how students and teachers may conceive the development of mathematics just as an evolutionary process. In doing that they loose the richness of the path of mathematical ideas that are lateral to the main stream of the development of mathematical concepts. Moreover we know that the intertwining and the reciprocal influence of internalist and esternalist factors is a powerful perspective for studying mathematical concepts and its development, as shown in the paper (Radford, 2006). Mathematics has changed during the time but has become also different in different countries and cultural contexts.

Ethnomathematics (see a product in Fig. 6) is a fruitful branch of research in education. It is about learning mathematics connected to other areas, to social and environmental problems (Joseph, 2003; Katsap, 2006). It lead to reflect on the fact that not only the European mathematics is the 'true mathematics'

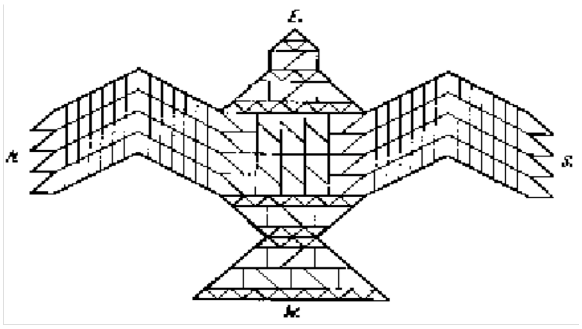

Fig. 6. The most elaborate altar from the Indian *Sulbasutras* (the first part probably was written in the 6th century B.C.). Many of the triangular and trapezoidal altars described in the *Sulbasutras* use then theorem of Pythagoras

Some external factors influence the daily work of researchers: relations among colleagues (well known 'spy stories' regarded 16th century Italian algebraists, see Fig. 7), salary (not ethically impeccable 'involvements' come from the fact that ancient and modern war requires a wide apparatus of mathematical knowledge), national policy pushed by the dominating class, see (Barnett, 2006; Swetz, 1987), etc. This is enough to confirm that context influences advancement of science.

Fig. 7. Italian mathematicians Niccolò Fontana ("Tartaglia"; 1499-1557) and Gerolamo Cardano (1501-1576)

MATHEMATICS VIEW

The ultimate aim of the book is to suggest a different mathematics view. Every chapter ends with a discussion about beliefs about the nature of mathematics, which are connected with the aspect treated in it. This part of the book regards factors that are not always made explicit in the classroom, but influence the personal relation with mathematics. We deem it is important to stimulate students' awareness on these factors. In the book the pictures and the related comments show unusual, but in our opinion more realistic, aspects of mathematics. As discussed above, mathematics:

- is an historical construction which is socially and culturally bounded, therefore different cultural context have produced different forms of mathematics;

- is used in many professions and jobs; is present in the everyday life; has epistemological and also psychological aspects which are intertwined (such as the role of error and its acceptance by individuals);
- has relationships with other disciplines; requires debate, communication and involvement and may also originate wish to investigate.

We briefly recall some beliefs widespread among students and ordinary people that were detected in our study (Demattè & Furinghetti, 1999). These are some of the beliefs considered in the book with respect to the content of every chapter: it is better if I remember rules by heart and I don't attempt to reason with my brain; when I solve a mathematical problem I know that there is only one exact solution; mathematics learnt in school has not a practical use; not everybody has a 'mathematical mind'; creativity is not necessary in mathematical reasoning; different topics, such as arithmetic, geometry, algebra, must be taught and learnt separately because they don't have any connections; in mathematics approximated results are incorrect and do not give useful information; in mathematics errors are absolutely negative experiences; mathematics doesn't depend on culture; I think that men have began to use the signs +, -, x, : before Christ; if I study alone (not with mates) I'll have better results in mathematics.

FINAL REMARKS

Only a few parts of the chapters have been administered in the classroom. The book is in progress and no student read it until now. It will be about 180 pages. After completing the work it is planned to propose some students to read at least a few pages, and to collect their opinions by means of questionnaires or an interviews.

At the moment some questions are waiting to be answered:

- How will readers consider the kind of mathematics which is described in the book? Will they establish connections with mathematics they learned at school or will they consider it an 'extraneous entity'?
- What beliefs could change learning the history of mathematics? What activities could be more useful?
- Learning history (in a broad sense) is also to remember fact and dates. What historical information could mathematics teacher require the students to remember? Could pictures create an opportunity to remember significant aspects of the history of mathematics?
- In our opinion, the citizen mathematics education requires new didactical choices. Could historical-epistemological analysis of mathematics replace some parts of traditional curriculum?

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APPENDIX. The structure of the book

In the book there is a preface explaining the aim and the rationale of the work and 30 chapters whose titles and some representative figures are shown below.

Legenda

E: *Chapters mainly concerning historical or **E**pistemological ideas.*

M: *Chapters over mainly concerning relevant **M**athematical topics.*

1. The first files of data (*M**)

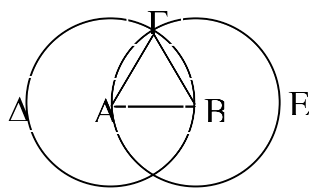
2. Mathematics for administering a Nation *(E)*

3. Is mathematics we learn at school ancient? (*E*)

4. How to write a number (*M*)

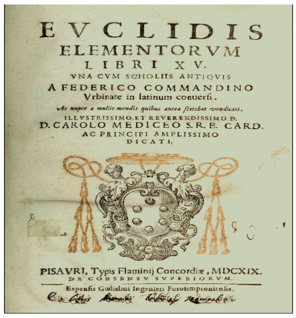
5. Algebra begins (*M*)

***6. Mathematics is full of errors* (E)**



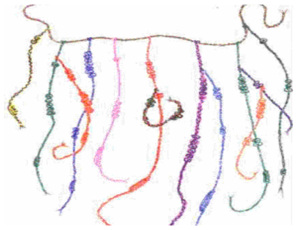
7. Pythagoras in China (*M*)

8. A model to imitate (*E*)


EVCLIDIS
ELEMENTORVM
LIBRI XV.
VNA CVM SCHOLIIS ANTIQVIS
A FEDERICO COMMANDINO
Vrbinate in latinum conuersi.
Ac nuper a multis mendis quibus antea scatebat vendicati.
ILLVSTRISSIMO, ET REVERENDISSIMO D.
D. CAROLO MEDICEO S.R.E. CARD.
AC PRINCIPI AMPLISSIMO
DICATI.
PISAVRI, Typis Flaminij Concordiæ, MDCXIX.
DE CONSENSV SVPERIORVM.
Expensis Gulielmi Ingenieri Forosempronitensis.

9. What is geniality? (*E*)

10. Does it depend on material we have? (*E*)



11. Mathematical knowledge doesn't "accumulate in layers" (*E*)

12. Recreational problems (*M*)

13. Does an authority hold knowledge? (*E*)

14. Mathematics is culture (*E*)

15. Masters of abacus (*E*)

16.Mathematics and trade (*E*)

17. Geometry for builders (M)

18. Mathematics and politics (*E*)


ALLO
ILLVSTRISSIMO
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IL S. COSIMO DE MEDICI,
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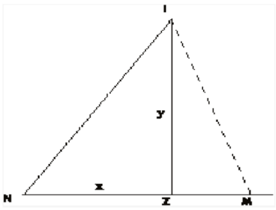
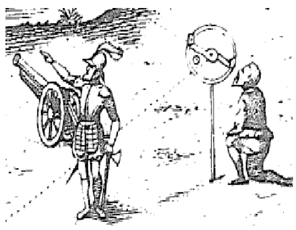
19. More recent than we think (*E*)

20. Is mathematics the same everywhere? (*E*)

21. Problems of paternity (*E*)

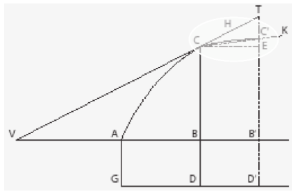


22. Mathematics and war (*E*)



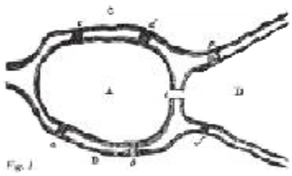
23. Let's bet everything (*M*)

24. Calculus (*M*)

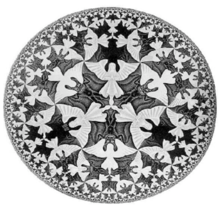


25. Mathematics and other sciences (*E*)

26. Geometry of position (M)

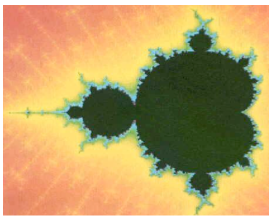


27. Beyond infinity (*M*)



28. Etnomathematics (*E*)

29. Past, present and future (*E*)



30. Imagine a mathematician (*E*)

STUDENTS' BELIEFS ABOUT THE EVOLUTION AND DEVELOPMENT OF MATHEMATICS

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The paper is an empirical study of students' beliefs about the history of mathematics. 26 students in an upper secondary mathematics class were exposed to a line of questions concerning the evolution and development of mathematics in the form of a questionnaire and follow-up interviews. In the paper it is argued that the existing literature on students' beliefs, in general, lacks a discussion of goals dealing with, for instance, desirable beliefs among students in order to provide them with a more coherent image of mathematics as a discipline. A couple of descriptions from the Danish literature and upper secondary regulations are provided as an example of such a dimension. The concrete student beliefs from the research study are evaluated against these descriptions.

KEYWORDS: History and epistemology of mathematics; students' beliefs and images; a goal-oriented dimension for students' beliefs.

INTRODUCTION

Beliefs about the history of mathematics is a topic which is touched upon from time to time in the literature on history in mathematics education, e.g. in Furinghetti (2007) and Philippou and Christou (1998). However, when scanning these samples, one soon finds that these concern the beliefs of in-service or pre-service teachers. Studies on students' beliefs about the history of mathematics seem to be rather poorly represented in the literature, if not altogether absent.[1] One reason for this that I can think of is that, in general, studies of beliefs in mathematics education are conducted with the purpose of improving mathematical thinking, learning, and instruction.[2] Beliefs, both cognitive and affective ones,[3] are investigated in order to identify the 'ingredients' which do or do not make students capable of solving mathematical tasks or teachers capable of teaching differently and/or more effectively. Certain beliefs are identified as advantageous in the learning of certain mathematical contents, the solving of related tasks, etc., and educational studies are then conducted on how to change already existing beliefs into these more favorable ones. In this sense beliefs are regarded as means – or *tools* – to achieve understanding in the individuals' constructive learning process. Only rarely is providing students or teachers with certain beliefs, e.g. by changing existing ones, about mathematics or mathematics as a discipline considered as a *goal* in itself. And when this is done, the term 'beliefs' is usually not used. Instead mathematical appreciation, mathematical awareness, or providing students with a more profound *image* of what mathematics is, are the words or phrases more commonly used (e.g. Furinghetti, 1993; Niss, 1994; Ernest, 1998).

It seems to me that the beliefs discussion in mathematics education lacks a goal-oriented dimension. A dimension which addresses students' mathematical world view and proposes and evaluates some desirable beliefs in order to turn students into more critical citizens by providing them with intelligent and concerned citizenship and with some *Allgemeinbildung* in general (Niss, 1994). That is to say, to provide students with a more coherent image of mathematics as a discipline, the influence of mathematics in society and culture, the impact of society and culture on mathematics, and the historical evolution and development of mathematics as a product of time and space, to mention a few of the more 'pressing' ones. Occasionally researchers will touch upon these issues in the form of personal opinions, e.g. in curriculum development. However, a dimension about 'beliefs about desirable beliefs' – meta-beliefs we may call them – can only be addressed properly if the meta-beliefs are articulated as such, i.e. as goals in themselves.

In this paper I shall first present some extracts from the 2007-regulations for the Danish upper secondary mathematics program and the Danish report on competencies and learning of mathematics, the so-called KOM-report, which may serve as such a goal-oriented dimension for students' beliefs. Especially I shall focus on students' beliefs concerning the history of mathematics. Secondly, I shall report on a piece of empirical research in which a number of students were asked about their beliefs concerning the evolution and development of mathematics.[4] Thirdly, these students' beliefs are analyzed and evaluated against the goal-oriented descriptions. The paper is ended with some final remarks and reflections on the presented empirical data and the larger research study which they are part of.

THE DANISH CONTEXT

Since 1987 history of mathematics has been part of the formal regulations for the Danish upper secondary mathematics program (see e.g. Fauvel and van Maanen, 2000, pp. 5-7), and with the newest reform and the present regulations of 2007 this part has become more dominant. Students are now expected to be able to "demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution", knowledge acquired through teaching modules on history of mathematics (Undervisningsministeriet, 2007, my translation from Danish).[5] The official regulations for the Danish upper secondary mathematics program of 2007 are to some extent based on the Danish report *Competencies and Learning of Mathematics*, the so-called *KOM-report*, (Niss and Jensen, 2002, title translated from Danish) where it says the following about history:

> In the teaching of mathematics at the upper secondary level the students must acquire knowledge about the historical evolution within selected areas of the mathematics which is part of the level in question. The central forces in the historical evolution must be discussed including the influence from different areas of application. Through this the students must develop a knowledge and an understanding of mathematics as being created by human beings and, in fact, having undergone an historical evolution – and not

just being something which has always been or suddenly arisen out of thin air. (Niss and Jensen, 2002, p. 268, my translation from Danish)

In the report, the focus of integrating history of mathematics is discussed in terms of a certain kind of overview and judgment which the students should acquire as part of their mathematics education.

> The form of overview and judgment should not be confused with knowledge of 'the history of mathematics' viewed as an independent subject. The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and subject to the motivations and mechanisms which are responsible for this development. On the other hand it is obvious that if overview and judgment regarding this development is to have solidness, it must rest on concrete examples from the history of mathematics. (Niss and Jensen, 2002, p. 68, my translation from Danish)

The 2007-regulations describe the "identity" of mathematics in the following way:

> Mathematics builds upon abstraction and logical thinking and embraces a long line of methods for modeling and problem treatment. Mathematics is indispensable in many professions, in natural science and technology, in medicine and ecology, in economics and social sciences, and as a platform for political decision making. At the same time mathematics is vital in the everyday. The expanded use of mathematics is the result of the abstract nature of the subject and reflects the knowledge that various very different phenomena behave uniformly. When hypotheses and theories are formulated in the language of mathematics new insight is often gained hereby. Mathematics has accompanied the evolution of cultures since the earliest civilizations and human beings' first considerations about number and form. Mathematics as a scientific discipline has evolved in a continual interrelationship between application and construction of theory. (Undervisningsministeriet, 2007, my translation from Danish)

Thus, when the students are to "demonstrate knowledge about the evolution of mathematics" etc., as stated in the academic goals of the regulations, one must assume that it is within the frame of this "identity" that they are expected to do so. Another way of phrasing this is to say that one purpose of the teaching of mathematics at the Danish upper secondary level is to mold the students' beliefs about mathematics according to the above description of identity. The purpose of including elements of the history of mathematics has to do with showing the students that mathematics is dependent on time and space, culture and society, that mathematics is not 'God given', that humans play an essential role in the development of it, etc., etc.

STUDENTS' BELIEFS ABOUT THE 'IDENTITY' OF MATHEMATICS

In the beginning of 2007, I conducted a questionnaire and interview research study of second year upper secondary students' (age 17-18) beliefs about the 'identity' of mathematics. A number of these questions had to do with the evolution and development of mathematics. In the following I shall present the students' answers to seven of these questions. All in all 26 students answered the questionnaire. The students' questionnaire answers have been indexed in the following manner: one<few<some<many<the majority<the *far* majority, a partition which roughly corresponds to the percentage intervals: 0-5%; 6-15%; 16-35%; 36-50%; 51-85%; 86-100%. Based on the questionnaire answers 12 students were chosen as representatives for the class in general, and these 12 students were interviewed about their answers. All quotes from the questionnaires and the interviews (the ones in blue) have been translated from Danish.

1. How do you think that the mathematics in your textbooks came into existence?

The majority of the students believe that the mathematics is due to people in history who have been wondering or been curious about something, and therefore attempted to explain what they observed. Many of the students who think so believe the people responsible for the mathematics are some special, wise persons and great "minds of ideas", a few mention Pythagoras. One student suggests that the ones responsible are "some very patient, half autistic people who have been wondering about the connections, rules, etc. between things, e.g. the angles of triangles, the lengths of the sides, etc." Some believe the mathematics to be an accumulation of experiences, observations, and experiments, possibly anchored in nature. A few of these emphasize the cumulative nature of mathematics. Others believe that mathematics was created because of a need, for instance in connection to trade or "in order to make things more manageable". The interviews provided no additional information.

2. When do you think it came into existence?

The majority believe that the mathematics in their textbooks came into existence "sometime long ago". The suggestions to exactly when are, however, many and different: "from even before da Vinci's time!"; "when the numbers were invented"; "when we began using Arabic numerals"; "way before it says in the books". Some points to antiquity and provide as argument that "the construction of, for instance, the pyramids must have required at least some mathematics". One of the more interesting answers goes: "Long, long ago it all began and since then it has continued. But I am confident that the development goes slower and slower, because you eventually know quite a bit."

Out of this majority of students, some share the perception that mathematics always has existed, or at least has existed as long as human beings have been around. One says: "Mathematics in general has existed since the dawn of time, but highly developed [mathematics] has only emerged within the last 200-100 years." Only one student believes the mathematics in the textbooks to be of a more recent date, and he is not afraid to fix this to "40 years ago".

In the follow-up interviews, events in the history of mathematics were occasionally fixed with some kind of accuracy, for instance, the beginning of mathematics to 4000-5000 years ago; Pythagoras to the first couple of centuries; and Fermat's last theorem to “the Middle Ages or something”. But only few students were able to do this. Whether this is due to lack of knowledge about history of mathematics or lack of knowledge about history in general, or maybe both, is not to say. Finally, one of the students seemed very strong in her belief that it was impossible to practice mathematics without the Arabic numerals. When asked why not, she answered: “the mathematics you do today, you wouldn’t have been able to do that...”

3. Why do you think it came into existence?

The majority of students believe that there was a need to have mathematics at one’s disposal. A few even talk about a necessity: “For example with constructions it has been important to be able to predict/calculate if, for instance, the walls can support the roof etc. Better to find errors on the drawing board than when the final construction collapses.” Many students mention the development of society and related aspects as the main causes. Some again mention that people have been wondering, been curious about something, and followed their ideas and impulses. One student ascribes the cause to “The will of God – or Big Bang, if you like.” In the follow-up interviews one student said: “Because people had too much time on their hands, for example, so they were given jobs as mathematicians.”

4. Are the negative numbers discovered or invented? Why?

In the answering of this question the class was divided in two approximately equal parts, one in favor of discovery and the other in favor of invention. The argumentations provided were quite different though. A few students believed the negatives to have been discovered in connection with or immediately after the positive numbers. Others believed that they always had been there, but that it might have taken some time to “learn to express them” or that people were “able to see it, but might have had difficulties explaining it”. Among the arguments for invention we find: “They are invented, I think, because you would get something wrong if they weren’t there”; “On the face of it, invented because you can’t have something which isn’t there”; “They are invented because you needed values smaller than 0”; “Think they are invented since it appears strange that a number all of a sudden should fall from the sky or something”. From time to time the same arguments were used for both discovery and invention: “Discovered. If we imagine a man who has bought a cow, but doesn’t have money enough, so that he owes money away, i.e. a negative number”; “Discovered. If you were in debt to someone, maybe”. One student plays it safer: “I’d think they were invented because almost all mathematics is invented, but at the same time also discovered.” The interviews provided no additional information.

5. Do you believe that mathematics in general is something you discover or invent?

The majority of the students believe that mathematics in general is something you discover. Only a few believe that it is something you invent. More students, though, believe that it might be a mix of the two. Many of those who believed that negative numbers were something discovered stick to this point of view for mathematics in general. A few of the answers are: “Discover. I don’t think you can invent mathematics – it is something ‘abstract’ you find with already existing things.”; “Discover. Because mathematics is already invented. What happens today is only that you discover new elements in it.” A lot of those who believed that negative numbers were discovered and a few of those who believed them to be invented now seem to think both: “Many things might begin as an invention, but afterwards they are explored and people discover new elements in the ‘invention’ in question”; “Both, [I] think that you discover a problem and then solve it by inventing a solution or applying already known rules of calculation”; “You discover formulas after having discovered relationships”. Some of those who believed negative numbers to be invented now believe mathematics in general to be discovered: “Mathematics is all over – in our society, our surroundings and in the things we do. Therefore I do not believe mathematics to be something you invent, but on the contrary something you discover along the way. Of course, it might be difficult to say precisely, because where is the line drawn between discovery and invention?” One of the answers touch upon the question of what mathematics ‘really’ is: “Good question... very philosophical. I think there are many different standpoints to this. I personally believe that it is something you discover. Numbers and all the discoveries already made are all connected. So for me it is more a world you enter into than one you make.”

In the follow-up interviews the student responsible for the last remark explained further: “Well, I see it as if mathematics is just there, like all natural science as, for instance, outer space. Outer space is there and now we are just discovering it and learning what it is. That’s what I think: It’s the same thing with mathematics.” When the remaining interviewees in favor of discovery were asked if the ‘exploration’ of mathematics corresponds to the exploration of the universe they all confirmed this belief. That is to say that they believed mathematics to always have existed, or as one student phrased it: “Mathematics has always been there, in the form of chemistry or something like that at the creation of Earth. And then we haven’t found out about it until later.” Or another one: “I think it has always been there, but I just think that the human beings are exploring mathematics more and more and are discovering new things.” One of the students who believed mathematics in general to be discovered thought that negative numbers were invented: “It is something you have made up because you had to. Well, it isn’t something written down somewhere from all eternity or from God, and which has just been there. It is something you invent because there is a need. If you invent a chair, right, then it is because you have a need to sit down.” As an answer this even links to the previous question of why mathematics came into existence.

6. Do you think mathematics has a greater or lesser influence in society today than 100 years ago?

The far majority of the students believe greater. This answer is in general based on the increased amount of technology in our everyday society. Answers as "definitely, more computer=more mathematics" and "everything develops and everything has to be high-technology" are often given. A few of those who believe that mathematics has a greater influence today also points to economic affairs as the reason, or that "the use of mathematics has become more advanced in our time". Some think that mathematics has the same influence today as it had 100 years ago, and only very few believe that the influence today is lesser. One of the more 'sensational' answers of the latter kind is: "No, I don't believe that, because even though we use mathematics a lot more in space etc. we have modern machines to do it."

The follow-up interviews to a large degree confirm the beliefs described above. To the deepening question of why a student found the influence today to be greater, she answered:

> Student: Because today you can, for instance, get an education at... or study mathematics at the university and things like that, and that you couldn't do a hundred years ago. [...]
>
> Interviewer: *So it is something relatively new that you can study mathematics at the university?*
>
> Student: No not new, but I do believe at a higher level. That is, you didn't know as many things back then as you do today.
>
> Interviewer: *And you couldn't get an education as a mathematician in the same way, you think?*
>
> Student: No.

The student who argued lesser influence due to the use of modern machines is also given the opportunity to expand on her view in the interviews. She finds, amongst other things, that mathematics appears less present because we rely on technical aids to a great extent, and because the use of mathematics is mostly about "pushing some buttons".

7. Do you think that mathematics can become obsolete? If yes, in what way?

To this question the far majority answered a clear no or that it appeared unlikely, for instance: "a proof is a proof" or "the basic things we build our mathematical development on are so used and tested that it won't become obsolete". Some provide a no with modifications: "Don't think that it can become obsolete, but that theorems/theories can be disproved and thereby provide a foundation for 'new mathematics'." Or more striking: "No, but there are probably some things which will not be used so much in the future: Such as vectors." Only a few answer yes or maybe. The follow-up interviews provided no additional information.

EVALUATING STUDENTS' BELIEFS AGAINST THE 'GOALS'

How do the above presentation of students' beliefs about the evolution and development of mathematics correspond with the goal-oriented description of overview and judgment in the KOM-report and the 'identity' of mathematics in the 2007-regulations? For example, are students able to "demonstrate [display] knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution"? Overall the students' answers to some of the questions appear rather diffuse, but let us take them from the beginning. As an answer to question 1 the majority seem to believe that mathematics has developed and evolved as a result of peoples' personal curiosity and wonder. Only few mention extrinsic reasons such as trade. In question 3, however, there is an agreement that mathematics has come into existence because of a need or even as a necessity. Thus, there is a slight incongruence between the answers of the majority to questions 1 and 3. Of course, one may interpret it as if the inner motivation and curiosity of people to get involved with mathematics have been turned on by outer circumstances, which for certain incidences in history would be correct. The fact that mathematics itself besides being driven by outer driving forces also is driven by inner driving forces (forces which not only concern the personal motivation of a single mathematician, but the intriguing problems within mathematics itself) is not an aspect which the students seem to be aware of. And concrete examples from the history of mathematics, in the form of the KOM-report's talk of "solidness" (cf. page 3), is not something which the students seem able to provide either. One student mentions the building of the pyramids as an example of the need of mathematics in older times, but if this answer is founded in a concrete knowledge about mathematics in ancient Egypt or if it is merely evidence of the student being able to think for himself is not to say. In any case he does not provide any concrete examples of Egyptian mathematics.

In the answers to question 2 there seem to be an agreement that mathematics is 'old'. One student implies that da Vinci is old and that mathematics is older than him. However, only very few are capable of providing years on the origin of mathematics as well as on concrete mathematical results. That some students believe that mathematics only could come into existence by aid of the Arabic numerals does not strengthen the interpretation that the students possess knowledge about the evolution of mathematics in interplay with historical and cultural events either.

Despite some discrepancies with the answers of question 4, the majority in question 5 give expression to the fact that they believe mathematics in general to be discovered. In a Danish educational context this may appear surprising since, as Hansen (2001, p. 71, my translation from Danish) puts it: "it is clear that the strong position of constructivism in school circles fertilizes the ground for a more radical constructivist perception of the entire nature of mathematics. Because of the pedagogical constructivism in schools, children and young people are likely to have difficulties believing in special existence of mathematical quantities, figures, and concepts." Of course there are students who are inclined toward a view of mathematics in general as something invented, but they are few in number. The majority gives expression to a Platonic stance. With the words of one of the students, it is “a world you enter into” – a world of ideas – where you explore the already existing mathematical objects in a similar way as we are exploring the Milky Way and the rest of the universe our planet is part of. Certainly such a view is bound to play down the creative side of mathematics as a human activity, and as a consequence perhaps also the KOM-report’s more ‘humanistic’ view of mathematics as something being created by human beings and not just suddenly having popped out of thin air (cf. page 2). On the other hand, the students seem to have a quite good understanding of the fact that mathematics today has a much greater influence in society than it did 100 years ago (question 6). Again it is computers and other technology that are given credit for this. The fact that students only pay scant attention to economic affairs and political decision-making, e.g. based on mathematical models, may be seen as a consequence of the invisibility of mathematics in society (Niss, 1994). One student touched upon this when she said that mathematics appears less present due to use of technology. Another example is the student who in question 2 believed that the development of mathematics was happening at a slower and slower pace and who in the interviews explained herself:

> Yes, but they just discovered more a long time ago, didn’t they? It isn’t very often you hear about someone who has discovered something new within mathematics, is it? Maybe it’s just me who isn’t enough of a mathematics geek to be told about it. But it just seems to me that nothing is really happening. Stu□ is happening more often within natural science: now they have found a method to see the fetus at a very early stage by means of a new type of scanning or something.

This student seldom hears about new discoveries in mathematics, even though she is exposed to the subject several times a week, therefore she believes nothing is happening. Besides this, her remark also touches upon one of the differences between mathematics and the natural sciences: just because mathematics now is able to prove Fermat’s last theorem or the Poincaré conjecture this is not something which will change our everyday or society neither tomorrow nor in 50 years (most likely), something which would be far more likely for discoveries in physics, chemistry, or biology – and to a larger extent for technology basing itself on these disciplines.

The students also seem to have an understanding of mathematics as a science which is not likely to become obsolete (question 7). This has to do with the ability of mathematics to most often include previous results as special cases in more general and abstract new constructs. Of course, one might mean many different things with ‘obsolete’, and the inclusion in theory building is only one aspect. Another way to see it would be to think of concrete applications of mathematics. In this respect, the history of mathematics provides examples of very old pure mathematics which suddenly finds its way into an application (e.g. the use of old number theoretic results in RSA cryptography), but probably the history contains even more examples of pieces of mathematics which until now have not, or maybe never will, find their way into applications. But the point is that one can never tell what will and what won't. In any case, one could have hoped that the students would have been able to provide some concrete examples of one or the other supporting their views – the only example was the student who thought that vectors were unlikely to be used for anything in the future. This supports the above mentioned lack of 'solidness' of the students' beliefs.

FINAL REMARKS AND REFLECTIONS

According to Lester, Jr. (2002, p. 352), Kath Hart at a PME conference once asked: "Do I know what I believe? Do I believe what I know?" Lester's version of this question is: "Do students know what they believe?" Furinghetti and Pehkonen (2002) argue that one should take into consideration both the beliefs that students hold consciously as well as unconsciously. But how to do this? Lester, Jr. (2002, pp. 352-353) sows doubt about some of the more usual methods for doing this: "I am simply not sure that core beliefs can be accessed via interviews [...] or written self-reports [...] because interview and self-report data are notoriously unreliable. Furthermore, I do not think most students really think much about what they believe about mathematics and as a result are not very aware of their beliefs." Thus, the results above must perhaps be viewed in this light. However, other researchers (e.g. Presmeg 2002) argue that questionnaires, interviews, etc. are perfectly well suited to access students beliefs about mathematics as long as the usual precautions, for example the interviewee trying to please the interviewer, are taken into account.

In the research reported in this paper, the students knew nothing about my personal viewpoints on the evolution and development of mathematics; they were not familiar with the descriptions in the KOM-report, or the 'identity'-description in the regulations for that matter. So it seems reasonable to say that none of these views could have affected the students' answers. Of course, they knew that the interviewer was a mathematician which might have led them to alter some of their views. Also, it is true that many students do not have a clear and conscious idea about their beliefs about mathematics, as Lester says. When asking the interviewed students to deepen or expand their questionnaire answers some of them would have trouble remembering what they answered, some would be puzzled about their own answers, and some would take on different viewpoints in the interviews than what they had expressed in the questionnaire. Especially the question of invention versus discovery was one that seemed to puzzle the students; often they would have difficulty in making up their minds. From an educational perspective, this is, however, the power of this precise question: that there is no correct answer to it. It is a matter of conviction, whether you are a Platonist, a formalist, a constructivist, a realist, an empiricist, or something else.

Thus, students will have to *reflect* about the question on their own in order to take a standpoint.

Especially reflection and the ability to perform reflection are considered to be major factors in changing beliefs (Cooney et al., 1998; Cooney, 1999). Thus, if the students who took part in the research presented above were to have their beliefs 'molded' or 'shaped' in such a fashion that they would fit the previously presented goal-oriented descriptions, then one way of doing this would be to set a scene which enabled them to perform reflections. In fact, the students' questionnaire and interviews reported above are an initial part of a larger research study, one purpose of which was to provide the students with classroom situations in which they were expected to work actively with and reflect upon issues related to, amongst other, the previously discussed aspects of the evolution and development of mathematics (questions 1 through 7). More precisely, these situations consisted of two larger teaching modules which the upper secondary class was to engage in over a longer period of time.[6] During and after the period of implementation, the changes in students' beliefs were attempted evaluated through more questionnaires and interviews but also by means of videos of classroom situations taking as the point of departure the 'initial' student beliefs as presented in this paper.[7] A comparison of the questionnaire and interview results presented in this paper, i.e. those from before implementing the modules, with the later research findings, those from during and after the implementations, will be presented in Jankvist (2009).

As a very final remark, I shall point to my own belief that reflections ought not only be considered as a means for changing existing beliefs, or creating new ones. A students' image of mathematics should include an awareness of mathematics as a discipline that consists of and gives rise to questions to which there are no correct answers (e.g. that of invention versus discovery), and for this reason the ability to reflect is equally important. That is to say that not only is the act of providing students with an image of, or a set of beliefs about, mathematics as a discipline a goal in itself, the act of making the students capable of reflecting about their images is a goal as well.

NOTES

1. An exception is a Danish study of Christensen and Rasmussen (1980).

2. A few examples are Schoenfeld, (1985) and Leder and Fortaxa, (2002).

3. I shall not here enter the discussion of defining 'beliefs'. I do, however, implicitly base my understanding of beliefs on the definition given by Philipp (2007).

4. The full questionnaire consisted of 20 questions covering historical and developmental, epistemological and philosophical, sociological, and more personal affective matters of mathematics. Questions 1 to 7 are a variation of these.

5. The word ‘demonstrate’ in Danish has a dual meaning; it may be used both as the word ‘prove’ and as the word ‘display’. Thus, students may only need to display knowledge.

6. Descriptions of and preliminary results from this research study may be found in Jankvist, (2008a) and Jankvist, (2008b).

7. E.g. beliefs on question 5 were evaluated by posing more specific questions relating to the cases of the two modules.

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USING HISTORY AS A MEANS FOR THE LEARNING OF MATHEMATICS WITHOUT LOSING SIGHT OF HISTORY: THE CASE OF DIFFERENTIAL EQUATIONS

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The paper discusses how and in what sense history and original sources can be used as a means for the learning of mathematics without distorting or trivializing history. It will be argued that this can be pursued by adopting a multiple-perspective approach to the history of the practice of mathematics within a competency based mathematics education. To provide some empirical evidence, a student project work on physics' influence on the development of differential equations will be analysed for its potential learning outcomes with respect to developing students' historical insights and mathematical competence.

INTRODUCTION

Fried (2001) argues that when history is used to teach mathematics the teacher must

> either (1) remain true to one's commitment to modern mathematics and modern techniques and risk being Whiggish, […] or, at best, trivializing history, or (2) take a genuinely historical approach to the history of mathematics and risk spending time on things irrelevant to the mathematics one *has to* teach. (Fried, 2001, p. 398).

Whig history refers to a reading of the past in which one tries to find the present.

The purpose of the present paper is to argue that this dilemma can be resolved by adopting (1) a competency based view of mathematics education, and (2) a multiple-perspective approach to the history of the practice of mathematics. Hereby, a genuinely historical approach to the history of mathematics can be taken, in which the study of original sources is also relevant to the mathematics one *has* to teach. To present some empirical evidence for this claim a student directed project work on the influence of physics on the development of differential equations will be analysed. The project belongs to a cohort of mathematics projects made over the past 30 years by students at Roskilde University, Denmark. Only one project is analysed in the present paper, but the reflections and discussions brought forward are based on knowledge about and experiences from supervising many of those projects.

First, mathematical competence and the role of history in a competency based mathematics education are presented. Second, a multiple-perspective approach to a history of the practice of mathematics will be introduced. Third, the chosen project work will be analysed and discussed with respect to specific potentials for the learning of differential equations within the proposed methodology. Finally, the paper ends with some conclusions and critical remarks.

MATHEMATICAL COMPETENCE AND THE ROLE OF HISTORY

In the Danish KOM-project (2000-2002) mathematics education is described in terms of mathematical competence. In this context mathematical competence means the ability to act appropriately in response to mathematical challenges of given situations and it can be spanned by eight main competencies (Niss, 2004). Half of them involves asking and answering questions in and with mathematics: (1) to master modes of *mathematical thinking*; to be able to formulate and solve problems in and with mathematics, i.e. (2) *problem solving* and (3) *modelling competency*, resp.; (4) to be able to *reason* mathematically. The other half concerns language and tools in mathematics: (5) to be able to handle different *representations* of mathematical entities; (6) to be able to handle *symbols and formalism* in mathematics; (7) to be able to *communicate* in, with, and about mathematics; (8) to be able to handle *tools and aids* of mathematics. In the discussion below, the possible learning outcomes of reading sources will be analysed with respect to these competencies.

History of mathematics is not one of the main competencies, but is included in the KOM-project as one of three kinds of *overview and judgement* regarding mathematics as a discipline. The first concerns actual applications of mathematics in other areas, the second, historical development of mathematics in culture and societies, and the third, the nature of mathematics as a discipline (Niss, 2004).

The KOM-understanding of the role of history in mathematics education has the honesty to history as an intrinsic part. In Danish secondary school this understanding of history is included in the curriculum (Jankvist, forthcoming). The objective of the present paper is to discuss in what sense such an understanding of history can be implemented in situations where the curriculum does not include history and does not assign time to teach history. Under such circumstances, history of mathematics is most likely going to play no role at all in the learning and teaching of mathematics unless it can also be used as a means to learn and teach subjects in the syllabus.

A MULITPLE PERSPECTIVE APPROACH TO HISTORY OF MATH

How can we understand and investigate mathematics as a historical product? One way is to think of mathematics as a human activity and of mathematical knowledge as created by mathematicians. This has been the foundation for many recent studies in the history of the practice of mathematics (Epple, 2000), (Kjeldsen et al., 2004).

To study the history of the practice of mathematics involves asking why mathematicians situated in a certain society, and/or intellectual context at a particular time, decided to introduce specific definitions and concepts, to study the problems they did, in the way they did it. In this line of thinking, mathematics is viewed as a cultural and social phenomenon, despite its universal character. Studying the history of mathematics then also involves searching for explanations for historical processes of change, such as changes in our perception of mathematics, our understanding of mathematical notions, and our idea of what counts as a valid argument.

A way of answering such questions is to adopt a multiple perspective approach (Jensen, 2003) to history where episodes of mathematical activities are analysed from multiple points of observations (Kjeldsen, forthcoming). The perspectives can be of different kinds and the mathematics can be looked upon from different angles, such as sub-disciplines, techniques of proofs, applications, philosophical positions, other scientific disciplines, institutions, personal networks, beliefs, and so forth.

How can this approach be brought into play to ensure the honesty to history, in a teaching situation where the teacher wants to use history as a means for students to learn a specific mathematical topic or concept? It can be implemented on a small scale, by having students read pieces of original mathematical texts focusing on perspectives that address research approaches or the nature and function of specific mathematical entities (problems, concepts, methods, arguments), in order to uncover, discuss, and reflect upon the differences between how these approaches and entities are presented in their text book and the former way of conceiving and using them. In such teaching settings, the students have to read the mathematical content of the original text as historians, using the "tools" of historians, and answering historians' questions about the mathematics. For such tools, see e.g. (Kjeldsen, 2009).

Through activities where students work with historical texts guided by historical questions, connections between the students' historical experiences of the involved mathematics and their experiences from having been taught the text book's version, can be created in the learning process. When students read historical texts from the perspectives of the nature and function of specific mathematical entities, they can be challenged to use other aspects of their mathematical conceptions in new situations. So, it is of didactical interest to analyse historical episodes of mathematical research with respect to their potential to challenge students' mathematical conceptions.

A HISTORY PROJECT: PHYSICS AND DIFFERENTIAL EQUATIONS

In the following, the student directed project work will be analysed with respect to how and in what sense the students' work with original sources provided potentials for the learning of differential equations – without losing sight of history.

The educational context: problem oriented student directed project work

The project report on physics influence on the development of differential equations was written by five students enrolled in the mathematics programme at Roskilde University (RUC). All programmes at RUC are organised such that in each semester the students spent half of their time working in groups on a problem oriented, student directed project supervised by a professor. The projects are not described by a traditional curriculum, but are constrained by a theme (Blomhøj & Kjeldsen, 2009).

The requirement for this project was that the students should work with a problem that deals with the nature of mathematics and its "architecture" as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the status of mathematics, its historical development, or its place in society gets illuminated.

Among the cohort of project reports, constrained by these objectives, this particular project was chosen, because the students happened to investigate differential equations, which are included in the core curriculum of advanced high school mathematics and mathematics and science studies in universities. Hence, the project work could be analyzed with respect to the issues addressed in the present paper.

Analysis of the project work: learning outcomes and the competencies

The students formulated the following problems for their project:

> How did physics influence the development of differential equations? Was it as problem generator? Did physics play a role in the formulation of the equations? Did physics play a role in the way the equations were solved? (Paraphrased from (Nielsen et. al., 2005, p.8)).

On the one hand, these are fully legitimate research questions within history of mathematics. They address issues about an episode in the history of mathematics seen from the perspective of how another scientific discipline influenced mathematicians' formulation of problems as well as the methods they used to solve the problems. On the other hand, these questions can only be answered by analysing the details of original sources that deal with this particular episode in the history of mathematics, studying how the differential equations were derived from the problems under investigation, how the equations were formulated, why they were formulated in that particular way, how they were solved and with which methods – issues which are also relevant for the learning and understanding of the subject of differential equations. Based on readings of three original sources from the 1690s, the students discussed these issues within the broader social and cultural context of the involved mathematicians, critically evaluating their own conclusions within the standards for research in history of mathematics. Hence, in this way of working with history in mathematics education history is neither Whiggish nor trivialized.

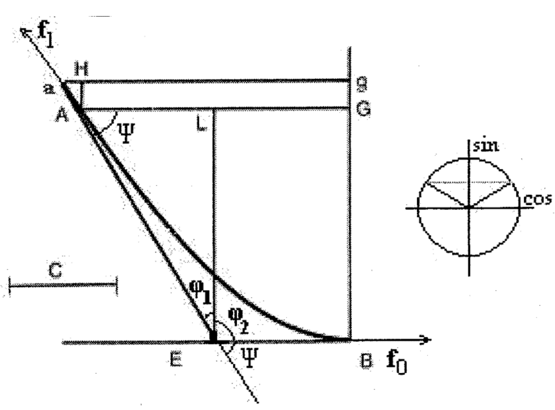
I will discuss three instances where the students – qua the historical work – were forced into discussions in which they came to reflect on issues that enhanced their understanding of certain aspects of differential equations in particular and of mathematics in general. The discussion will end with a short presentation of some of the learning outcomes with regard to the eight main mathematical competencies.

1: Johann's differential equation of the catenary problem. The catenary problem is to describe the curve formed by a flexible chain hanging freely between two points. The students read the solution that Johann Bernoulli presented in his lectures on integral calculus to the Marquis de l'Hôpital, supported by English translations of extracts (Bos, 1975). Bernoulli formulated five hypotheses about the physical system that, as he claimed, follow easily from static. For the students, of which none studied physics, to derive these assumptions was the first mathematical challenge in reading Bernoulli's text: "we had to derive most of them ourselves. We use 18 pages to explain what Johann Bernoulli stated on a single page" (Nielsen et. al., 2005, 19).

Below is one of the extract of Bernoulli's text (Bos, 1975, 36) that the students read. As can be seen from the text, Bernoulli used the five hypotheses to describe the catenary and the infinitesimals *dx* and *dy* of the curve geometrically and derived an equation between the differentials. The figure was produced by the students and is similar to a figure in Bernoulli's lecture, except from the sine-cosine circle.

Assuming these results, we go on to find the common Catenary as follows: Let *BAa* be the required curve, with its lowest point at *B*; its axis, the vertical line through *B*, is *BG*. The tangent to the curve at its lowest point is the line *BE*, which will be horizontal. Let the tangent at any other point *A* be *AE*. We draw the ordinate *AG* and a line *EL* parallel to the axis. Let $BG = x$, $GA = y$, $Gg = \mathrm{d}x$ and $Ha = \mathrm{d}y$.

Because the weight of the chain is distributed evenly along its length we may put that weight equal to the length of the curve *BA* which we call *s*. Thus since an equal and constant force will always be required at the point *B* (by hypothesis (iv)) whether the chain *BA* is made longer or shorter, let this force be of magnitude *a*, expressed by the straight line *C*. Let us now imagine the weight of the chain *AB* to be concentrated and suspended at the point *E* where the tangents *AE* and *EB* meet. Then (by hypothesis (ii)) the same force is required at *B* to support the weight *E* as was previously required to support the chain *BA*. Indeed (by hypothesis (v)) the ratio of the weight *E* to the force at *B* is the same as the ratio of the sine of the angle *AEB*, or of its complement angle *EAL*, to the sine of the angle *AEL*, that is the ratio of *EL* to *AL*. Therefore, whatever position on the curve is taken for the fixed point *A* (the curve, by hypothesis (iii), being always the same) the ratio of the weight of the chain *AB* to the force at *B* is the same as the ratio of *EL* to *AL*, that is $s:a = EL:AL = AH:Ha = \mathrm{d}x:\mathrm{d}y$, and inversely $\mathrm{d}y:\mathrm{d}x = a:s$.[1]



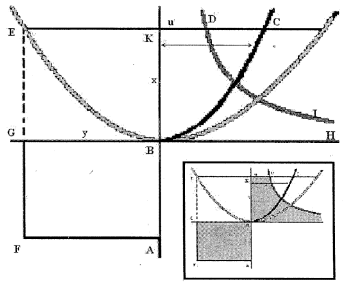
In their report, the students went through Bernoulli's text and filled in all the arguments. They were not familiar with this way of setting up differential equations from scratch so to speak, so the mathematization of the physical system was a major challenge for which they needed to consult some textbooks on static and to combine the physics with mathematical results about triangles and the sine-cosine relations.

Bernoulli's arguments do not meet modern standards of rigour and that created cognitive hurdles for the students. Didactical, it is important to identify such hurdles because they create situations where the students, during their struggle with understanding the mathematical content of the original text, can be challenged to reflect upon the differences between our modern understanding and the one presented in the source, thereby enhancing their own understanding of the concept of, in this case, differential equations and the mathematical techniques and concepts underneath. A concrete example of this is Bernoulli's use of the infinitesimal triangle. In the text above he used similar triangles, to argue that $s:a = dx:dy$ but, as the students pointed out in their report, *a* does not lie on the tangent but on the catenary. Bernoulli also used the infinitesimal triangle later in the lecture, when he reformulated the differential equation derived above, using that $ds = \sqrt{dx^2 + dy^2}$. Again – as pointed out by the students – *ds* is a part of the catenary, not the hypotenuse of a right angled triangle.

This mixed use of geometrical arguments and infinitesimals in deriving and reformulating the differential equation was very different from the students' text book experiences of differential equations. The fact that Bernoulli's method worked in this particular case, despite its lack of rigour, provoked a discussion among the students and their supervisor (the author) about Bernoulli's use of the infinitesimal triangle and his use of the infinitesimals, *dx* and *dy*, as actual infinitely small quantities. This made the students focus more systematically on the differences between now and then, questioning, at first, why we need to define a differential quotient as the limit (in case it exists) of difference quotients, then analysing the situation again to understand why Bernoulli's method worked fine for the catenary, and trying to picture situations where it would go wrong. This is an incidence where connections were created between the students' historical experiences and their experiences from modern mathematics which challenged them to examine their own understanding of the involved concepts. Through these discussions, the students built up intuition about infinitesimals and awareness about the reasons behind the construction of our modern concepts. Major differences were the lack, in the seventeenth century, of the concept of a function, of a limit, and the formalised concept of continuity. In this project work the historical texts provided a framework for discussions among the students and with their supervising professor, about what constitute the concept of a differential equation, and how we can read meaning into it. Through these discussions, which were triggered by the historical texts, the students came to reflect upon the concept of a differential quotient and the meaning of a differential equation on a structural level that went beyond mere calculations and operational understanding of the concepts. This is an example of what Jahnke et. al (2000) calls a reorientation effect of studying original sources.

2: Johann's solution of the catenary differential equation. Through some further manipulations Bernoulli reached the following formulation of the equation for the catenary $dy = adx/\sqrt{x^2 + 2ax}$ which he used to construct the curve geometrically. This puzzled the students and initiated discussions about, what it means to be a solution to a differential equation.

Let the normals *AK* and *GH* be drawn, meeting in *B*. Take *BA* = *a* and describe an equilateral hyperbola *BC* with vertex *B* and centre *A*. Now construct a curve *DI* with the property that everywhere *BA* is the middle proportional between *KC* and *KD*, that is such that *KD* = *aa*:√(2*ax* + *xx*). Now draw a parallel *AF* and take the rectangle *AG* equal to the area *HBKDI*. Prolong *DK* and *FG*, then their intersection point *E* will be on the required curve.[1]



As can be seen from the above extract (Bos, 1975, 41), Bernoulli interpreted the integral geometrically, as the area below a curve. The students added an illustration of this in their figure, as can be seen above, with the two shadowed areas which are not present in Bernoulli's figure. This way of solving the equation by constructing the curve forced the students into discussions about conceptual aspects of solutions to differential equations. It made them articulate what constitute a solution in our modern understanding, an articulation that does not automatically manifest itself from solving differential equation exercises from modern textbooks. In order to follow Bernoulli's construction, the students were challenged to think about and use integration differently than they would normally do when solving differential equations analytically. They were also forced to use the properties of the curve represented geometrically which they felt as a challenge. They were used to using the direct relationship between the analytical expression of a function and the coordinate system, to produce a graph. Here they went "the other way" and had to think of the curve as being represented by its graph instead of its analytical expression. Historically, they realised that what is understood by a solution to a differential equation has changed in the course of time.

3: Different solution methods of the brachistochrone problem. The brachistochrone problem is to describe the curve of fastest descent between two points for a point only influenced by gravity. Jacob and Johann Bernoulli published different solution methods to the problem in 1697. Johann Bernoulli interpreted the point as a light particle moving from one point to another. By using Fermat's principle of refraction, he derived an equation for the brachistochrone, i.e. the cycloid, involving the infinitesimals dx and dy. Jacob Bernoulli considered the problem as an extremum problem using that, since the brachistochrone gives the minimum in time, an infinitesimal change in the curve will not increase the time.

The differences between Johann's and Jacob's solution of the brachistochrone illustrated for the students the power of mathematics. Johann's solution was tied to the physical conditions of the problem and could not be generalised beyond the actual situation, whereas Jacob's solution was independent of the physical situation and could be used on different kinds of extremum problems. Through the historical texts on the solution of the brachistochrone, the students experienced the characteristics of the nature of mathematics that makes it possible to generalise solution methods of particular problems. Thereby, they were able to understand why Jacob's method could generate new kinds of questions that eventually led to a new research area in mathematics, the calculus of variations, and why Johann's could not. For a presentation of the historical problem of the brachistochrone in a didactical perspective, see Chabert (1997).

Development of mathematical competencies. In the discussions above of episodes where the students through their work with the original sources used other aspects of their mathematical conceptions in new situations and discussions, some learning potentials regarding differential equations and the mathematical concepts underneath have already been emphasised, especially in the discussion of the students' work with Johann Bernoulli's text on the catenary. A more systematic analysis of the students' report with respect to the KOM-report showed that the students, in their work with the historical texts, were challenged within seven of the eight main competencies. The students' awareness of the special nature of *mathematical thinking* (1) was especially enhanced in their comparison of Johann's and Jakob's solutions of the brachistochrone as discussed above. The students' *problem solving* (2) skills were trained extensively and in different areas of mathematics. As mentioned in the discussion of their work with Johann's solution of the catenary problem, the students' had to fill in a lot of gaps in order to understand Johann's results. Each of these gaps required that the students derived intermediate results on their own about similar triangles using trigonometry, and solved mathematization problems. Through their work with understanding the Bernoulli brothers' mathematization of the physical problems, parts of the students' *modelling* competency (3) were developed. The competency to *reason* (4) in mathematics was developed in all those parts of the project work where the students tried to make sense of the original sources by means of their own mathematical training and knowledge. (5) *Representations*: As exemplified in the discussion of the students' work with Bernoulli's construction of the solution to the differential equation of the catenary, the students were challenged so work with a representation of the solution to the differential equation that is different from the analytical representation given in modern textbooks. In the report, the students also solved the differential equation analytically and compared the analytical representation with Bernoulli's geometrical one. During their mathematization of the five hypotheses from static that Bernoulli took for granted, the students were trained both in working with different representations and in using the mathematical language of *symbols and formalism* (6). This competency was especially developed in the students' work with the two original sources on the brachistochrone problem in their struggle to understand Johann's mathematization of the path of the light particle and Jakob's use of the minimising property of the brachistochrone. The writing of the report (90 pages) in which the students, through a thorough presentation and analysis of the original sources, answered their problems for their project work within the historical context, developed their competency to *communicate* (7) in, with, and about mathematics in ways that go far beyond what normal exercises in solving differential equations requires. The competency to handle *tools and aids* (8) was not represented.

SOME CONCLUSIONS AND CRITICAL REMARKS

Based on their studies of the original sources and relevant secondary literature, the students concluded that physics did function as problem generator in the early history of the development of differential equations and played a decisive role in the derivations of the equations describing the catenary and the brachistocrone. They further concluded that physics played a significant role for Johann's solutions of both the catenary and the brachistochrone problem, but not for Jacob's solution of the brachistochrone problem. Jacob's arguments were not linked to the physical system; hence his method could be transferred to other problems of that type. This became the beginning of the calculus of variations. The students did not move beyond this in their project, but it is interesting to notice that the calculus of variation later became central in physics, providing an important feedback in the opposite direction.

The analysis of the chosen project has shown that, *if* we adopt a competency based view of mathematics education and evaluate learning outcomes not with reference to standard procedures and lists of concepts and results, but with respect to how and which mathematical competencies, the students have been challenged to invoke, and thereby develop, and *if* we let the students work with the history of the practice of mathematics studied from specific perspective(s) that address(es) significant issues regarding the mathematics in question, then history can be used as a means to teach and learn core curriculum subjects without losing sight of history.

The above claims are further supported through analyses of other historically oriented mathematics projects that have been performed by students at RUC. A project on the history of mathematical biology, where the students read an original source of Nicholas Rashevsky on a mathematical model for cell division is treated in (Kjeldsen & Blomhøj, 2009) and analysed with respect to learning outcomes regarding deriving and understanding the general differential equation of diffusion, the students' understanding of the integral concept, and development of the students' modelling competency. Other examples of projects with substantial learning outcomes of core mathematics, in university mathematics education, are "Paradoxes in set theory and Zermelo's III axiom", "What mathematics and physics did for vector calculus", "Generalisations in the theory of integration", "Infinity and "integration" in Antiquity", "Bolzano and Cauchy: a history of mathematics project", "The real numbers: constructions in the 1870s", and "D'Alembert and the fundamental theorem of algebra". In the present paper focus has been on how history can be used for the learning of core curriculum mathematics without trivializing it or using a whiggish approach to history. The learning outcome of the above history projects can also be analysed with respect to *Mathematical awareness*, as explained by Tzanakis and Arcavi (2000), which includes aspects related to the intrinsic and the extrinsic nature of mathematical activity. These projects can then also be seen as empirical evidence for some of the possibilities history offers as referred to by Tzanakis and Arcavi (2000, 211). With respect to the KOM-report these aspects relate to the three kinds of *overview and judgement*.

It can be raised as a critic that only certain perspectives of the history are considered, and that e.g. to gain insights into historical processes of change, episodes from different time periods need to be studied. In the above project work, the students did not experience the historical process of change, but they did experience that the understanding of the involved mathematics in the 17th century was different from our understanding. The students did not solve a huge amount of differential equations through their historical studies, and they did not learn to distinguish between different types of differential equations.

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WHAT WORKS IN THE CLASSROOM - PROJECT ON THE HISTORY OF MATHEMATICS AND THE COLLABORATIVE TEACHING PRACTICE

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This paper describes the project that was undertaken in the South East of England, and which aimed to introduce the history of mathematics at the primary and secondary level. The project was conducted through collaborative teaching practice (peer based network of teachers collaborating on research, planning, teaching in teams, and assessing the outcomes of lessons) and was based on the premise that the history of mathematics can improve both the motivation and attainment when used as a contextual background in the teaching of mathematics at this level.

THE PROJECT BACKGROUND

The project described here was one of the first few projects awarded the support by the National Centre for Excellence in the Teaching of Mathematics (founded in June 2006). Aims of the project were to:

- Introduce the history of mathematics into everyday teaching in order to
  - Encourage students to begin making the connections between mathematical topics
  - Increase interest and motivation by setting the problems in historical context
  - Enrich mathematical understanding through historical explorations
  - Assess the role of the history of mathematics in setting the new curriculum
- Introduce collaborative teaching practice as a model of continuing professional development, at the same time adopting an inquiry-led learning approach to the lesson development thus raising issues about
  - Teachers learning with pupils (simultaneously in some cases) and the effects this may have on his or her professional role
  - Training preparation for teachers in an inquiry-led learning environment.

The answers to these questions will be provided in this paper in two-fold ways: through the personal reflections of teachers who participated in the project, and through a synthesis and explanation of methods used throughout the project. The latter is provided as a way of suggesting the model of continuing professional development for teacher groups and networks wishing to introduce the historical element into the teaching of mathematics through collaborative practice.

The project began in September 2006 and was completed in September 2008 with a national conference held at the London Mathematical Society at which experiences of the teachers involved were disseminated among the mathematics education community. Over the course of the project three secondary schools, with a total of fifteen teachers (two of whom were science specialists but taught mathematics to lower ability groups), and three primary schools with a total of three teachers have been involved. More than 450 pupils have been involved in the project at various times, spanning the age range between ten and fourteen (English Key Stages 2 and 3) and covering all ability ranges.

The project has been conceived and led by the author of this paper, and, as already mentioned, was supported by the National Centre for Excellence in the Teaching of Mathematics (UK). In the second year of the project the British Society for the History of Mathematics provided financial and organisational support; the University of Plymouth Centre for Innovation in Mathematics Teaching provided the training for all involved teachers in the principles of collaborative teaching practice, and the British Society for the History of Science provided extra funds for the final conference celebrating the project. An additional private consultant has been involved in the project in the second year, offering support in the matters of teacher training and the uses of the history of mathematics in development of mathematical pedagogy.

The new curriculum for England and Wales

The recent changes in the National Curriculum, and the new approach taken by the Qualifications and Curriculum Authority (QCA) introduced a certain amount of freedom for teachers, teacher teams, and consortia of schools to develop their own syllabus in all subjects. The modernising of the curriculum is driven by the need to take into account local needs and needs for different types of vocational training. One of the more positive aspects of this development may be seen in the fact that the local provision of education will have a degree of freedom (not yet defined), and that personalised learning, project based work and mentoring will all have a big role to play in this new vision of education. This opens a valuable opportunity for teachers to demonstrate that mathematics, like any other creative pursuit, is an area where exciting and useful contributions can still be made – both by teachers and by pupils. As such, the introduction of the historical element in the mathematics syllabus, although not sufficiently developed in the quote that follows, offers the ***possibility*** of developing teaching strategies which do not necessarily provide only historical context, but use the history of mathematics as a tool for discovering facts and exploring mathematical techniques. The new curriculum states that the students should recognise the 'rich historical and cultural roots of mathematics':

> Mathematics has a rich and fascinating history and has been developed across the world to solve problems and for its own sake. Students should learn about problems from the past that led to the development of particular areas of mathematics,

appreciate that pure mathematical findings sometimes precede practical applications, and understand that mathematics continues to develop and evolve.[1]

Since the completion of the project, and based on the recommendations following from the project report, measures are being taken by the Joint Mathematical Council (UK) to define the ways in which history of mathematics can and should be deployed to help shape the future development of the curriculum, and the teacher pre-, and in-service training development and provision.

The current challenge now facing English teacher-training institutions will be to address the imbalance between the desire to introduce the historical element to the teaching of mathematics and a lack of the formal teaching in the subject area for the serving teachers. The project described can therefore, give a valuable insight into the types of issues facing teachers in this situation, with a view of defining some benchmarks on which it would be possible to base a programme of in-service training in the history of mathematics.[2]

METHODOLOGY, ACTIVITIES, DATA

Collaborative Teaching Practice and the History of Mathematics

The project has been pursued by practicing teachers with various degrees of experience in the teaching of mathematics (not all of whom are subject specialists), and therefore the question arose of how to create a professional learning environment which would be able to contain all levels of experience and mathematical ability in order to support their participation. Of major interest was the possibility of introducing a model of continuing professional development based on a set of principles which could be replicated elsewhere and which would help teachers develop a range of techniques, and introduce a new element which could help them structure their own learning at the same time as structuring their teaching programme.

We chose the model of collaborative teaching practice as one which would offer opportunities for teachers to develop their subject knowledge through research into the history of mathematics. Collaborative teaching practice was developed in different countries as far back as the 19th century (most prominently Japan, but recently also in the United States and England) and is sometimes also closely linked and/or referred to as 'lesson study'.[3] The collaborative teaching practice that was part of the described project as a way of peer-discussion and collective teaching tool was based on the simple cycle of planning - researching - sharing resources - teaching collaboratively - and finally assessing the outcomes of a lesson.

At the core of this envisaged professional learning model stood a belief that the interest and personal development can only be achieved in those situations and environments where the professionals themselves find an area of research they would like to pursue further.

Various mathematics educators have seen the different roles the history of mathematics can take through its introduction into the education of mathematics teachers - Freudenthal (1981) for example conceived it as giving a background to the teachers' mathematical knowledge, while others concentrated on offering a possible pathway to the deepening of teachers' reflection capabilities through an in-depth study of the development of mathematical concepts through history (see Arcavi, Bruckheimer, & Ben-Zvi, 1982, 1987; Swetz, 1995). One of the approaches, developed by Hsieh and Hsieh (2000), and Philippou and Christou (1998a, b) dealt with using the history of mathematics as a particular tool and context to develop beliefs and attitudes in mathematics.

The benefit of the use of history of mathematics however, in the context of the described project, can be best seen on the influence in which it created an opportunity for a ***focus*** of cooperation and collaboration as well as an impetus for the creation of a ***conceptual landscape*** which offered opportunities to teachers to develop their individual interests.

This highly individualist approach to the continual professional development of teachers can increase their subject knowledge and enable them, through the modern technologies, to share their experiences and knowledge with mathematics teachers and students from around the world. Our agreed aim was to adopt a creative and individualistic ethos in teaching, providing ample opportunity for bringing the history of mathematics alive to the present generation of school children. Eventually, in practical terms, the defined foci were enlarged to include, apart from the collaborative teaching practice and the individual research, the creation of a networking platform in the form of web-quests[4].

The inquiry-led learning

The inquiry-led learning is, on occasions, redeployed in contemporary practice with the individualised workshop-type of learning.[5] The basis of the model the project team adopted rested on the modern interpretation of the heuristic method of teaching geometry; one in which pupils are encouraged to discover intuitively some geometrical truth without the resource to the available knowledge to begin with. At an appropriate time, the historical element is being introduced, showing pupils how others dealt with the same or similar problem, thereby

- enhancing the learning process by making connections
- increasing interest and motivation by setting the problem in context
- enriching mathematical understanding through historical context.

The project team was aware of the problems that inquiry-led teaching may contain should it be deployed without the full understanding of the possible drawbacks - unstructured or poorly structured learning environment, the difficulty of leading students to 'discover' complex theories, and the difficulty some teachers may have in adjusting to such a learning environment. We however opted to explore this type of environment helped by the peer network and by the view that learning in a 'mobile' world must change to incorporate not only technology but the ideas of learning that the pupils/children already posses by the time they come to attend school.

Inquiry-led learning also raises a number of questions for the preparation of lessons, teacher training, and finally, curriculum development - all aspects of the described project. Through the work conducted in the latter part of the project, during the winter/spring term of academic 2007/8, it was possible to make some conclusions regarding these questions.

Teachers' learning in an inquiry-led learning environment, and the collaborative teaching practice

The inquiry-led learning as developed through this project grew organically from the collaboration with similary-minded colleagues. The successful outcomes were produced in those instances in which a few necessary prerequsites were fulfiled - existence of full professional trust and exchange of information and knowledge had to be devoid of all performance management in participating groups of teachers. The peer network, on the other hand, offered plenty of opportunities for exploring the areas of improvement instead. Critical friends were deemed to be colleagues working within smaller groups, and the involvement of the higher education (Plymouth) and national (NCETM) institutions added a dimension to this process through validation and provision of a postgraduate course.

Collaborative teaching practice was described in the teacher reflections thus:

> The students appreciated the teachers cooperating between themselves and being more relaxed and focused on learning rather than discipline.
>
> It (this project) has certainly been a huge milestone in my professional development. Firstly, it has shown me the true value of collaborative teaching and the focus on the 'learning' rather than the 'teaching'. Secondly, it has made me question why I am teaching what I am teaching, and how to help the children answer the 'why' do we do this questions by giving them relevance and meaning to the maths. My next milestone experience will be to embed this into my teaching and more crucially into the teaching of my colleagues.

History of mathematics and the development of the curriculum

In the description of the other aspects of this project it is described how the history of mathematics helped shape the building of the professional learning environment which then spilt over into the classroom. Historical dimension, apart from earlier mentioned benefits (see pages 1-4) was also important for teachers in terms of their involvement with the whole-school issues:

> The maths becomes 'embedded' in the culture and life and is not seen as something totally dry and devoid of meaning. This also changed the perception of mathematics in my department… (by a science teacher)
>
> There is a large scope in my school to bring about change in the mathematics curriculum and I am hoping to introduce an element of the History of Maths into the

curriculum. 'Using and Applying Mathematics' is the common strand that is across the whole maths curriculum, and my experience on the project is that practical maths (in and out of the classroom) is a powerful medium by putting the children in the shoes of mathematicians from history so they can appreciate the 'why' and not just the 'how'.

OUTCOMES - STRUCTURING THE SELF-REGULATORY CONTINUING PROFESSIONAL DEVELOPMENT THROUGH COLLABORATION AND RESEARCH

The project showed how the history of mathematics can set the 'scene' and act as a catalyst in creating a professional learning environment as well as giving a structure to endorse inquiry both in the student and in the teacher. In mathematics, this dimension is or can be, added to any such particular conceptual landscape. This should work with the teaching of any branch of elementary mathematics, but mostly so in the case of geometry. As the subject matter itself deals with understanding of space and properties of spatial elements, this in turn helps and underlines the development of competency in the building of a conceptual landscape of interrelated mathematical ideas. One may say that at the ontological level the building of the networks of concepts underlines the exploratory process of building the structure of learning mathematics, thus making the learning of geometry a truly multi-dimensional knowledge manifold.

The history of mathematics and the process of reorientation

As Furinghetti has shown (2007) some teachers tend to believe that the style of mathematics teaching they were affected by or exposed to must be reproduced in their own practice. In the case of the described project, this was most evident in the attitudes of teachers who were non-specialists in the subject. Furinghetti showed that the history of mathematics context allows for an exploration of topics in a new light and hence helps teachers construction of teaching sequences. While this was one of the added benefits of introducing the history of mathematics into the collaborative practice, we were also aware of the uses of history of mathematics in teaching, therefore allowing us to explore the various roles the history of mathematics can take in the classroom practice.

Whilst the history of mathematics in teacher education programmes has been described at some length by Furinghetti (2007), Schubring (Schubring et al., 2000), and Heiede (1996), little has been so far written about the in-service training of practicing teachers in this regard. This project aimed to begin the task by making a sketch of the possible influence the history of mathematics can have on in-service specialist and non-specialist mathematics teachers.

Therefore one of the project's aims became to try to introduce what Furinghetti (2007) calls 'reorientation':

> …the learners involved in the process … are forced to find their own path towards the appropriation of meaning of mathematical objects.[6]

In this context, the acquisition of meaning was attempted through exposing beliefs about, and the partial understanding of, the concept in question with the new, 'foreign' meaning:

> A meaning only reveals its depth once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closedness and one-sidedness of these particular meanings.[7]

This was found to be particularly effective in two instances:

- the teachers' understanding of mathematical concepts developed through the process of familiarisation towards meaningful context, followed by cognitive expansion, leading in turn to formal definitions, and thereby enabling specialist and non-specialist teachers to acquire:
    - greater competency in the subject matter
    - understanding beyond the basic comprehension of the type 'this is how things work'
    - development of interest to further deepen the knowledge and understanding of the mathematical concept and its relation to other concepts in the knowledge landscape.
- teachers began structuring their understanding of concepts by developing an ability to switch between modes of thinking and behaviours attributable to learner vs teacher, therefore:
    - deepening their own understanding of the process of discovery and learning
    - refocusing from 'how it works' to 'why' as well as '***how*** and ***why*** did ***they*** do it'.

In short, one of the teacher testimonies illustrates these described process thus:

> … I was… astounded (by)… the depth there is in so many topics we have covered through this project. It has rekindled interest in mathematics in me; students find it interesting as well.

Scaffolding knowledge for non-specialist mathematics teachers

An increasing body of research shows that inquiry-based-learning helps create an environment in which the teacher may be required to act in manifold ways.[8] These manifold roles of a teacher relate to the theory of 'Knowledge Manifolds', in which teachers are 'promoted' from teacher/preacher to teacher/consultant and teacher/resource type of roles. Naeve (2005) defined the 'Knowledge Manifolds' as 'linked information landscapes (contexts) where one can navigate, search for, annotate and present all kinds of electronically stored information'.[9] Such open information landscapes have developed with an exponential speed since the founding of Wikipedia (domain launched only in January 2001), and rest on fundamental principles of communal and self-governance in the same way in which Naeve suggests future 'teaching landscapes' will develop. This theory is in concordance with the network theories of knowledge as much as it is with the theory of 'mobile learning'. The described project opted to further explore in practice such approach to teaching and learning in which teachers are as much learners as their pupils by making parallels between the sets of teachers with the sets of pupils. Some teacher reflections addressing this particular aspect are:

> This project has developed my skills to be able to find resources and to try to relate things to the history.
>
> Research was good for subject knowledge; because of the historical content, it widened our own perspective about mathematical topics, and gave us time to find about something in more depth.
>
> Historical element shows you the different aspects of something in more depth; it allows for 'scaffolding' of the knowledge and easier transference to children. The historical element can also offer easier focus.

Furthermore, Naeve's (2005) approach to knowledge which he identifies as that consisting of 'efficient fantasies' and learning as that consisting of 'inspiring fantasies' has a lot to offer in the context of creating a learning environment in which both teachers and students discover new facts and exchange ideas in a more elaborate, creative, and yet mathematically sound ways. Naeve's description of fantasy has a lot to offer in terms of initiating a process of learning not only in the here and now, but one that draws upon the initial interest in the 'fantasy' and how it (the fantasy) occupies a mind of a learner for a longer period of time, offering a prolonged urge to find ever increasingly new content about a subject matter. Teachers from the project spoke often about these 'fantasies' as most important in the initial stages of introducing a new mathematical topic or concept. The length of this paper does not, unfortunately, allow for further analysis on the subject matter in more depth.

What the conclusions teachers made however, agrees with Naeve's suggestion that the education process consists in

> …exposing the learner to inspiring fantasies and assisting her/him in transforming them into efficient fantasies.[10]

While Naeve somewhat exaggerated the view of the traditional 'learning architectures' being exclusively teacher-centric and consequently his concept of knowledge 'pushing' rather than knowledge 'pulling' may be lacking in subtlety, his intention to shift the focus onto the system of initiation into an interest field, whilst at the same time offering the system of skills to equip a learner with a set of tools to undertake the task of discovery and learning is at the centre of all: 'collaborative', 'flexible', and 'personalised' learning concepts.[11]

So far, as in the case of Mariotti (2000), the focus on developing strategies to initiate 'learning fantasies' has been on the pupils. In the new type of learning environment, one in which 'knowledge pulling' rather than 'knowledge pushing' is taking place, teachers and pupils are learners and communicators of insights into mathematical facts at the same time, interchanging roles at different levels. From the experience of our project it became clear however, that some of the roles of the learner and some of the roles of the teacher are interchangeable, whilst others remain strongly rooted in the

a) evolutionary roles and

b) social roles these two groups represent.

Use of IT in developing CPD strategy in relation to the history of mathematics

> Much of the schooling is about learning to access parts of the cultural record and to manipulate them using the tools of external working memory such as writing and mathematical notation.[12]

Two aspects of the use of IT were deemed a necessary part of the development of the project:

1. using ICT to support the creation of narratives
2. using ICT to support the exploratory aspect of the learning.

To satisfy the former, a collection of web-quests[13] is being developed - self contained websites for each of the lessons taught and studied as part of the project. These fulfil many roles, one of most important being the development of the base of knowledge in the history of mathematics which is multi-dimensional and usable not only for the purpose of one lesson but available for re-use and individual study by pupils.

It is an undisputable fact that one of the most basic aims of all education is to instruct a learner into the sets of conventions. In the case of mathematics however, there is a danger that the convention may be confused with the invention - for example the fact that one (1) is not a prime number is often seen by a non-specialist teacher as a convention rather than a mathematical fact. The children too can have a difficulty in distinguishing the two important but entirely separate concepts, which can lead to the misinterpretation of all mathematics as a field consisting entirely of the various compilations of conventions. The historical narrative in this respect also offers a role in distinguishing the two. The process may be enriched by the narrative purposefully designed to satisfy the students' need to encompass all natural phenomena in a self-contained world of ideas thus also satisfying the mind's ability to form a dialogue between "particular 'episodic' events and the general 'theoretic' models".[14]

This however, offers the greatest challenge to a working teacher - finding the time and reliable resources which they can use to enable them to:

- construct such a narrative
- construct the resources or create an environment to support the inquiry-led learning.

Although in this respect the field of exploration widens considerably when the history of mathematic is offered as such one possible landscape, the necessity for the discovery and re-discovery of mathematical facts remains the question that needs to be addressed.

CONCLUSION

Although no external evaluation had taken place to date, the internal, self-evaluation, concluded that this was an invaluable opportunity for all teachers involved in the project in terms of re-awakening their interest in the subject and increasing their self-awareness on their abilities in terms of subject knowledge, pedagogy and ability to conduct academic research. Additionally, teachers identified acquisition of skills in terms of ability to envisage their own CPD landscapes through building 'knowledge patches' and increased ICT competencies as further valuable benefits of their involvement in the project.

The nature of learning is a constantly changing environment, in which learners are often ahead in terms of their technological competencies than their teachers. The knowledge content does not move at such a great speed, but it's presentation and availability is something that often lacks sophistication in the eyes of the learner. In mathematics this is sometimes more often apparent than in subjects such as literature or history.

Mathematics learning has to gain an enormous amount from developing landscapes of knowledge patches that students can tap into through and because of their interests and abilities. This project began the process of enabling the teachers to be able to start developing these landscapes in collaborative environment, and having for a focus the wealth of resources that the history of mathematics has to offer.

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[1] Page 4 of the QCA Mathematics Curriculum, accessed 20th March 2008,<<http://curriculum.qca.org.uk/subjects/mathematics/keystage3/index.aspx>>.

[2] As this paper was being completed, the new module in the history of mathematics was being developed at the Open University UK, aimed at anyone interested in the history of mathematics.

[3] See Lewis (1995), Lewis and Tsuchida (1998), Stigler and Hiebert (1999), and more recently Fullan (2004), (2005).

[4] Self-contained websites offering materials for the study of particular mathematical topics. First webquest from this project is available from http://www.webquests.mathsisgoodforyou.com/.

[5] The inquiry-led learning's main proponents were John Dewey (1859-1952) and Martin Wagenschein (1896-1988). The main view they propagated was that the understanding must come before knowledge, and some of their ideas form the basis of the 'constructivist' idea of learning. The anti-proponents suggest that the data collected over the past half-century does not suggest that inquiry-based methods actually work. See Kirschner, Sweller, and Clark, (2006) for the latter view.

[6] Furinghetti (2007), 113.

[7] Bakhtin (1986), 7, as reported by Radford, Firinghetti, and Katz (2007), 108.

[8] Naeve describes these roles as that of "*knowledge cartographer* [who] constructs context maps, the *knowledge librarian* [who] fills the maps with content, the *knowledge composer* [who] combines the content into customised learning modules, the *knowledge coach* [who] cultivates

questions, the *knowledge preacher* [who] provides answers, the *knowledge plumber* [who] routes questions and the *knowledge mentor* [who] provides a role model and supports learner self-reflection." Described in Naeve (1997).

[9] Naeve (2005), 6.

[10] Naeve, (2005), 4.

[11] All part of the national strategies on 'Every Child Matters', 'Personalised Learning' and 'Extended Schools'. See related sections at the <<http.//www.standards.dfes.gov.uk>>.

[12] Donald (1991), 329.

[13] Web-quests are self-contained websites which can mutate over time, and incorporate all elements for the study of a topic from introductory remarks to worksheets, and the possibility of submitting work for assessment. For further information and a collection of examples see << http://webquest.org/>>.

[14] Shaffer & Kaput (1998), 101.

INTUITIVE GEOMETRY IN EARLY 1900S ITALIAN MIDDLE SCHOOL

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A distinction between intuitive and rational geometry formally appeared in the Italian school program after the Italian unification of 1861. This distinction, that is not just an Italian issue, loosely corresponds to the points of view also adopted in the current geometry school programs both at a primary (6-10 and 11-14) and at a secondary (14-19) level. It is not difficult to define rational geometry: Although it has been approached with various methods, it is undeniable it arises from Euclid's elements. On the contrary, it is more complex to give a definition of intuitive geometry and to understand in which way it leads to rational geometry. This paper will illustrate the interpretation given to intuitive geometry by the school programs and by the many authors of textbooks at the end of 1800s and beginning of 1900s in Italy. This analysis can help to discuss today's curricular issues.

Key – words: Intuitive geometry – curriculum – history – school books.

INTRODUCTION

The term *rational geometry* first appears in the Italian school programs in 1867, few years before the complete Italian reunion, which occurred in 1871. A school reorganization brought in Euclid's *Elements* as *the* geometry textbook aimed to teach the subject in the Gymnasium-Lycée.[1]

In 1881, intuitive geometry comes to life to be taught in the first three years of the Gymnasium (the "lower Gymnasium" corresponding to the present middle school). Previously, geometry was not part of the school programs for students in this age.

As we will see forward, intuitive geometry was explicitly introduced as a propaedeutic subject to let students better understand the rational geometry studies.

It was not just an Italian issue to make a distinction between intuitive and rational geometry. Although with a different interpretation, references to intuitive geometry can be found also in the German and English literature of the same period (Fujita et al., 2004). In the textbooks of Treutlein (1911) and Godfrey & Siddons (1903), intuitive geometry - still propaedeutic to rational geometry – is identified with the ability to perceive a shape in a space, partially aiming to provide the basic elements which explain the real world, and partially aiming to develop logic skills. Accordingly, Fujita et al. describe intuitive geometry as "the skill to 'see' geometrical shapes and solids, creating and manipulating them in the mind to solve problems in geometry". This definition does not surely correspond to the characterization given by the Italian legislators at the end of the 18th century.

It is not difficult to give a definition for rational geometry. The term *rational*, opposite to intuitive, is meant to refer to any aspect of the logical and theoretical organization of the geometry (Marchi et al. 1996); although rational geometry can be approached in different ways, Euclid's Elements always remain at the foundations of this subject. On the other hand, it is more complex to define intuitive geometry and to analyze the way it is linked to rational geometry. Many researchers in math's education tackled this issue; a remarkable example is given by Van Hiele levels theory (cfr. Cannizzaro & Menghini, 2006).

The lack of a formal definition and of a detailed tasks' description of intuitive geometry caused continuous role changes in the Italian school programs. We believe it is important to discuss and analyze the reasons and the episodes which led to the introduction of intuitive geometry in the Italian school programs in the period between the 19th and the 20th centuries.

SCHOOL PROGRAMS

In 1881, elementary geometry and geometrical drawing were introduced in the first three years of the Gymnasium. An earlier intuitive experimental approach was considered a good help for students to overcome the difficulties caused by rational geometry and by the logical deduction of the Euclid's textbook. Geometrical drawing too should contribute to overcome these difficulties. Intuitive geometry had to

> give to youngsters, with easy methods and, as far as possible, with practical proofs, the first and most important notions of geometry, …useful not only to access geometry, but also to let the students desire to learn, in a rational way, the subject throughout the Lycèe.

Moreover, rational geometry is postponed to the Lycèe, skipping the two years of the higher Gymnasium, in order to avoid all the difficulties caused by its study.

Three years later, the new minister, following a suggestion of the mathematician Beltrami, abolishes the study of intuitive geometry from the lower Gymnasium and anticipates rational geometry to the 4th year of the Gymnasium. This decision was a consequence of a lack of clear boundaries, and of the fear that teachers could not emphasize in the right way the experimental-intuitive nature of geometry being tied to the traditional logic-deductive aspect of rational geometry (Vita, 1986 p.15).

In the following years, only few changes were introduced concerning the beginning of the study of rational geometry - which could be anticipated to the third year of the G*ymnasium* - and the learning approach to Euclid's books. According to Vita (1986, p.16), "the oscillation reflects a clear didactic anxiety and the desire of finding the most psychologically adequate time to teach *The Elements* by Euclid, with all its logic-deductive layout, to the 13-15 years old pupils".

In 1900s a new program was broadcast: intuitive geometry was restored in lower Gymnasium, but, to prevent past problems, the program included only elementary notions such as the easiest geometrical shapes vocabulary, the rules to calculate lengths, areas and volumes and also basic geometrical drawing. Some instructions specify that the new studies "were propaedeutic to rational geometry". Moreover, they underline that these new studies are "a review and an expansion of the notions acquired by the students at the elementary school", and require a practical approach, amplified by the teaching of geometrical drawing. As regards rational geometry, the new programs gave more freedom in the choice of the textbook, as long as it follows the "Euclidean method" (cfr. Maraschini & Menghini, 1992).

INTUITIVE GEOMETRY TEXTBOOKS IN EARLY 1900S

Since the program dated 1881 was effective for a very short period, we cannot find textbooks of intuitive geometry in those years. They appear right after 1900, instead. One of the first is the textbook by Giuseppe Veronese (1901). In Veronese's book we can easily notice the effort made to follow the ministerial program[2], considering the main properties of the geometrical shapes using the simple observation, rather than the intuition. Veronese wants to deal only with "those shapes that have an effective representation in the limited field of observation". Initially, not even the straight line, the plane and the unlimited space are matter of his dissertation, given that they need an abstraction process. Furthermore, Veronese believes it is dangerous to introduce concepts that will need to be amended at some stage in the higher studies.

In the *Peliminary Notions,* Veronese gives examples of *objects* (table, house..) and of their *properties* (colour, weight..). Material points (grains of sand) lead to the abstract concept of point, and material lines (a cotton thread) lead to the abstract concept of line, which is defined, both with practical examples (a pencil line) and as a *linear set of points* (an anticipation of what students will find in his textbook for the Lyceé).

All the authors of intuitive geometry books of this period introduce the straight line using the idea of a stretched string, and explain later on the way it can be drawn using a ruler. Veronese 'surrenders' to the temptation of stating in a more abstract way the reflexive, symmetric and transitive properties of the equality relation for the segments. Afterwards, he explains that the congruence of the segments can be verified using a ruler or a compass. Here is an example on how the classical distance axiom is interpreted from an observative point of view:

> Assuming that the extension of the field of observation is appropriate, it is possible to verify that: On a straight line r, given a point A and a segment XY, two segments exist CA and AB having the same direction and length of XY. The axiom can be proved using a piece of paper marked with a segment of the same length of XY, and sliding it along the line r in the direction showed by the arrow C ---> A ---> B X Y (p.9).

The textbook includes only one simple proof. After the definition of symmetric points about a given point O (central symmetry), Veronese states the following:

> The shape symmetric to a line about a given point is another line.
>
> Let ABC be a line and A'B'C' the shape opposite to ABC about a point O. Using a compass, or copying the shape AOB on a piece of drawing paper and turning the paper

up side down so that OA corresponds to OA' and OB to OB', we can verify that the point C' is on the line identified by B' and A'... (p.13).

Veronese, to avoid infinity, states that two lines are parallel when they are symmetric about a point, and explains how to manually verify that two lines are parallel (p.14). He lists elementary definitions for triangles, quadrilaterals, other polygons and for the circle without stating any property of these shapes.

In all his book, Veronese includes simple drawing exercises, meant to be done by hand (to draw a dotted line, to duplicate a segment marking some corresponding points, to draw symmetric shapes using a specific point as centre of symmetry). He introduces only at the end of the book some geometrical constructions, "aiming to improve, with the practice, the intuitive perception of the geometrical shapes, whose structure will be later analyzed using logical proofs". The chapter, describing geometrical constructions (of a triangle given three sides, of the bisector of an angle and other more complex constructions) which are not linked to the previous chapters, tacitly uses theorems never illustrated earlier in the book (especially those concerning the congruence of triangles). Some instructions precede this chapter, explaining how to execute a clear drawing and how to test the quality of rulers, squares, rubbers and pencils. Although Veronese made a good work keeping the manuscript simple, we have to note that no intuitive or rational effort is requested from the student.

Frattini's textbook (1901) has a structure which is similar to book by Veronese. He only gives less importance to the preliminary notions, more weight to the properties of polygons, and he also adds some minor practical proof. In the book's introduction, Frattini underlines that a "geometrical truth" exists, and it comes from "an immediate observation of the things, which is the essence of the intuitive method". In Frattini's book, lines and planes are unlimited from the beginning and parallel lines characterization changes to the one that everyone knows (parallel lines never meet). Lets us see the characteristics of some of his proofs.

There is exactly one perpendicular line through a given point to line on a plane (p.21). Let us bend a plane, imagine an immense piece of paper, and shape right angles so that one folding follows the line we want to draw the perpendicular to, and the other folding must include the point where the perpendicular passes through. Let us reopen the paper, it will be possible to see the trace of the perpendicular through the point and the line.

To state that "the sum of the three angles of any triangle is equal to two right angles (p.29)", Frattini uses the classic proof, based on the congruence of alternate angles. This congruence, anyway, is introduced without a proof ("the student can find a reason"). Veronese does not write about this property, not even about its consequences.

The diagonals of a parallelogram mutually bisect (p.33). Suppose we cut out the parallelogram from a piece of paper, we would have, then, an empty space which could be filled either placing the parallelogram back in the same position or placing the angle A, marked with an arc, on top of the equivalent angle C, the side AD on the equivalent

> side CB and the side AB on CD. In this way the diagonals of the shape, though upside down, would be in the previous position, the same for their crossing point. The two segments OC and OA would switch their positions: this means they are the same length.

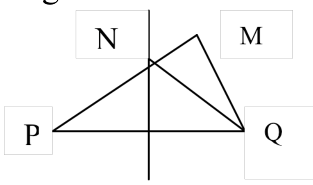
With regards of geometrical constructions, they are positioned, as well as in Veronese's book, at the end of the book. However, Frattini tries, when it is possible, to explain them, using the properties of polygons.

In 1907, a book by Pisati was published, slightly dissenting, in the preface, the programs' structure and stating as follows:

> it seems proved that, in lower middle school, it would be a big mistake to leave the formal aspect of the subject completely apart. Pupils' intellect, in the previous years of their life, has a formal nature….. Certainly, intuitive teaching of geometry is not easier than formal teaching;

In fact, his book starts stating the concepts such as axiom, postulate, theorem, corollary and problem. In his textbook, we can find explicit theorems and proofs. In example, Pisati introduces the reflection about a line and proofs that:

> Theorem - All points on the axis of a segment, and no other points, are equidistant from the endpoints of the segment.
>
> Proof. The first part of the statement follows from the properties of the axis of symmetry. To proof the second part, we see that, when the point M does not belong to the axis of the segment PQ, one of the line segments MP, MQ must intersect the axis (see fig.). Let us suppose that MP is the segment intersecting the axis and N the point of intersection. Consequently, we have NP=NQ. Thus MP = NP + NM = NQ + NM. Since NQ + NM > MQ; we have MP > MQ.



The proof of the theorem which states that the sum of any two sides of a triangle is always greater than the third one is justified by considering the line as the shortest distance between any two given points. This contested metric definition of the line, which was also used by Frattini, will never be used again in any geometry textbook for the secondary Italian school. The theorems proved by Pisati, allow him to explain all geometrical constructions stated at the end.

The title "intuitive geometry", which is not in Pisati's book anymore, completely disappeared from middle school textbooks, and will only reappear with Emma Castelnuovo's book in 1948.

FURTHER DEVELOPMENTS

In 1905, the Minister Bianchi feels the need to remind to "escape from abstract statements and demonstrations" adding, on the other hand, to use "simple inductive reasoning" to teach the "truths required by the school programs". In 1923, the reform made by Gentile turned the clock back. In the first three years of the *Gymnasium*, geometry studies "must only aim to keep alive all geometrical notions that the pupils have learnt at the primary school and to properly fix in their memory the terminology". Therefore, there are less requirements than in the provisions dated 1900. Amongst the books published right after the reform Gentile, we have to mention Severi's textbook (1928) which includes a preface by the Minister of Public Education. In spite of the good comments given in the preface, it is difficult to say that the book follows the school program guidelines. Over the years, middle school geometry had lost its experimental-intuitive nature, or even its terminological function, becoming more and more rational. Textbooks were almost independent from the school programs –which were in fact very brief and without any particular didactic connotation. The book by Severi is not surely an exception (although his book for higher school has always been appreciated for the experimental approach to theorems). It includes many theorems (also those regarding the angles at the centre and the angles at the circumference of a circle), with the most traditional proofs, except for using transformations (rotation and symmetry) as a support to the proofs and for avoiding the word "theorem".

In 1936 and 1937, a couple of reforms introduced only minor variations, which allowed some simple deductive analysis in the lower *Gymnasium*.

In 1940, the first three-year of the *Gymnasium*, of the Technical school ad of Istituto Magistrale[3] are unified to form the middle school. With reference to geometry, although its intuitive nature was confirmed, it was suggested to emphasize the evident properties "by means of several suitable examples and exercises, which, sometime, can also assume a demonstrative connotation…". So, we can find a bigger change compared to the small ones introduced in 1936: the purpose is to start from an intuitive way of thinking to go towards a more abstract logic nature.

An interesting book by Ugo Amaldi (1941) followed this reform. Amaldi completely stopped the process of "rationalization" of geometry. His textbook is similar to Frattini's book, but it contains some new important changes: measurements and geometrical constructions are not illustrated in separate chapters but they are integrated with the other parts of the book, providing a useful didactic tool. We find many figures and references to real life (i.e. an opening door gives the idea of infinite planes all passing through the same straight line, paper bands illustrate congruent segments…), which were completely disappeared in the meantime. So, given the instructions to draw the axis of symmetry of a segment using a ruler and a compass, Amaldi suggests to check the construction folding the paper and verifying that the circumferences, used for the construction, overlap. To know the sum of the angles of a triangle, he suggests to cut the corners of a triangle drawn on paper, to place them one next to the other and to check that they form an angle on a line. Similarly, he suggests cutting and folding techniques to verify the properties of quadrilaterals.

At the end of the world war in 1945, a Committee, named by the Allied Countries, deliberated some programs which were later adopted by the Italian Minister. The middle school program reverted to practical and experimental methods, but the methodological guidelines for the higher *Gymnasium* are particularly interesting: it is suggested to leave more space to intuitive skills, to common sense, to the *psychological and historical origin* of theories, to physical reality, ... to use spontaneous *dynamic* definitions which better fit the intuitive method.

Vita observes that "unfortunately these suggestions appear to be disjointed from the school programs that do not show any peculiar innovation". An innovation is, indeed, represented by the book of intuitive geometry by Emma Castelnuovo (1948). In her book, E. Castelnuovo follows in Amaldi's footsteps, using drawings, pictures, cross references to reality and integration of constructions and measurements. In addition to this, her book, for the very first time, interacts with the student, not only to let him follow a logic deduction or a proof but also she also raises questions in his mind.

> What is the meaning – you would question – of the statement that there is only one line passing through two distinct points A, B? How can the contrary be possible? It is true: it is not possible to imagine two o more distinct lines passing through A and B. It is possible, however, to draw with a compass several circles passing through two points…

The book starts with paper folding, and goes on with ruler and square constructions. As Amaldi does, she reuses the idea of the stretched string to introduce the properties of segments and straight lines; a method already used by Clairaut, who was Castelnuovo's inspiration. Simple tools are made-up, as a folding meter to show how to transform a quadrilateral into a different one, and to analyze the limit situations.

CONCLUSIONS

Our analysis clearly shows the difficulty to find an equilibrium between the notions that a pupil is supposed two learn, and the notions which he can accept by means of a non rigorous argumentation. It could seem that geometrical constructions were a real nuisance for early 1900s authors, due to their hidden theoretical content. Around the twenties, the problem seemed to be overcome by amplifying the rational aspect of geometry. It was only in the forties that the books of Amaldi and Emma Castelnuovo succeeded in the attempt to integrate constructions in the intuitive geometry textbooks, reducing their number and their technical aspect. We have to admit that most authors, starting from Veronese and Frattini, as Amaldi and Castelnuovo, perceived the need to reduce the dissertation: books are concise, authors are not eager to complete all topics, on the contrary, everybody tends to prefer a specific aspect of the subject.

Anyhow, the very aspect that seems to be relevant for approaching geometry in a real intuitive way is the active learning role of the student. Programs tried, several times, to deny this role, and it has been interpreted in different ways by authors. Emma Castelnuovo foresees and opens the door to the use of concrete materials.

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[1] Secondary education was divided into a first and a second level. To cover classical secondary education, a law of 1859 had introduced the Gymnasium and the Lycée - The Technical School and the Technical Institute were set up for technical secondary education.

The Gymnasium and the Technical School were preceded by four years of primary school. The Technical School thus covered the same age range as the present-day middle school (11–14) while the Gymnasium lasted for five years and hence included the first two years of high school followed by three years of Lycée.

[2] Index: preliminary notions; line; plane; equal shapes; plane polygons; circle; perpendicular lines and planes; polyhedra; cone – cylinder – sphere; sum, difference and measure of segments and angles; measure of segments and angles; surface areas, volumes; exercises. Drawing tools; basic constructions; Line, plane and unlimited space.

[3] Training school for primary school teachers.

THE HISTORICAL AND COGNITIVE DEVELOPMENT OF CALCULUS IDEAS

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SUMMARY

This contribution is a theoretical research about possible parallelism between the historical development and the cognitive development of mathematical ideas.

Epistemological obstacles identified in the history may be considered as candidates to be obstacles in the analysis of teaching and learning process. Thus, an important question is whether the kind of epistemological history obstacles are the same found during the learning at school. In that sense, the application of genetic method in teaching, presupposes that, for the understanding of a particular concept, the student has to repeat roughly the historical process that has evolved to the current formulation of the concept.

Key words. The genetic method - cognitive development - historical development - epistemological obstacles - epistemological breaks

INTRODUCTION

The use of epistemology helps us to maintain a vision of extrinsic objects taught, returning a historical vision to these objects, as opposed to the traditional teaching that tends to present them as universal objects. It is customary at the university, teachers address the teaching of science and mathematics in particular, usually based on finished facts, in which student does not get the notion of debate or controversy. It is also common to consider the construction of scientific knowledge as merely cumulative. In that sense, the historical reconstructions can provide students an understanding about the changing nature of science. While the development of mathematical concepts in the classroom can not be parallel, in general, to the historical development, it is conceivable, in a transposition didactic process, to go on the "stages" of the historical process.

Authors increasingly emphasize the need for teachers to value the importance of offering a vision of mathematics through the historical aspects that have influenced the construction of knowledge. In this sense, Garcia Cruz (1998) introduces the bachelardiano model (Bachelard, 1938), which is based on the idea of scientific change, within which there are three well-defined categories in the context of epistemology:

epistemological barriers: They are ingrained ways of thinking, old structures, both conceptual and methodological, impeding the progress of scientific knowledge. (Brousseau 1976, in Artigue, 1992: 197). It is accepted that the new knowledge is based, to some extent, in a process of rejection, contradicting the education model that sees learning in a linear way, adding new knowledge to the former. The presence of breaks in learning is normal. To learn ways of definitive knowledge, it´s necessary to go back, and progress in learning requires some sort of rejection of what has temporarily been an engine of progress.

epistemological breaks: In general terms, they are the ways in which scientific knowledge contradicts the ideas or beliefs that come from foremost primary knowledege, intuitive and common sense. The break that occurs between two different scientific concepts, both for a given knowledge to a specific methodology, it's also considered an epistemological break. Any break involves overcoming the corresponding barrier.

Epistemological acts: They are the mechanisms by which epistemological obstacles are overcome and breaks with the old concepts are favored, causing corresponding changes and improving the scientific vision about reality. Within these mechanisms the use of the history of science plays a vital role, especially when attempting a reconstruction of the processes that has conditioned the progress of scientific knowledge.

EPISTEMOLOGICAL OBSTACLES IN INTEGRAL CALCULUS TEACHING

The context

The introduction of the last 80 years reform has been very slow in America, and in Argentina in particular. On the one hand, formalism and theoretically still dominate the curriculum of calculation, and on the other hand, the new literature has a glaring lack of structuring. In this aspect, Michele Artigue (2000), referring to this problem in France, believes that the difficulties are clearly noticeable when reading most recent textbooks, where the status of objects, and the notions of assertions, are vague and unclear. These views characterize also our literature. The formal definitions have been rejected, replaced by expressions more or less accurate in a more common language, the theorems are accepted based on some explorations and they are not always mentioned as they could be. In terms of Artigue, it gives the impression that consistency induced by logical coercion of mathematical knowledge has disappeared, without another way to replace the solid consistency.

One of the didactic phenomena which is considered essential in the teaching of Mathematical Analysis, is the "*algebrización*", that is: the algebraic treatment of differential and integral calculation.

Artigue (in Contreras, 2000) expresses this fact in terms of an algebraic and reductionist approach to the Calculus, which is based on the algebraic operations with limits, differential and integral calculus, but it treats in a simplistic way the thinking and the specific techniques of analysis, such as the idea of instantaneous rate of change, or the study of the results of these reasons of change.

Contreras (2000) explains that when a teacher explains a particular mathematical concept and does not address, or does so on a superficial level, the typical problems of analysis, sliding into algorithmic postures easier to manage and evaluate, produces a genuine break of contract teaching, called by Brousseau “Topaze effect"(Brousseau, 1986).

Regarding the concept of integrate, we find it appropriate to make some considerations about the misconceptions in the students:

A) very often, students identify integrate with primitive. For these students, it’s not involved any convergence process nor any geometric aspect in the calculation of the integral. It is therefore a purely algebraic process, so that astudents can learn different methods of integration, and even apply them to calculate, with some fluency, and at the same time, not being able to solve the calculation of an area or to study it as a Riemann sum.

B) Students identify definite integrals with the rule of Barrow, even where it’s no proper to apply it. Such it’s the case of essential discontinuity functions in the range considered.

C) A third problem stems from the lack of association between the definite integral and the analysis of convergence.

D) It’s not integrated the concept of area with the concept of definite integrate. It is likely that students have noticed that there is a relationship between both, the definite integrals and the area, but teachers don’t work with their students on these issues, so that, it remains a purely algebraic interpretation of the integral. Students use the formal-algebraic context instead of visual-geometric one, simply, because they haven’t integrated them. (Llorens and Santonja, 1997).

It is clear that the epistemological obstacles identified in the history must be considered as candidates to be obstacles in the analysis of teaching / learning process. To be considered genuine epistemological barriers of education, it is necessary to test its resistance to teaching appropriate interventions. Thus, an important question is whether the kind of epistemological history obstacles are the same found during the learning at school or university. In that sense, the approach in terms of epistemological obstacles is usually associated with a search in the history of mathematics, in search of significant and fundamental problems and to an organization of the teaching process epistemologically more appropriate than the usual.

As part of an epistemological-historical analysis, we believe it is important to promote the construction of comprehensive knowledge about the techniques used to study their convergence, applications such as area between curves and volumes; both, for, as against the prior algebraic knowledge of students, which justifies the use of counterexamples that show the fallacies of over generalization, so breaks and necessary reconstructions may be promoted.

Obstacles related to the existence of the area and volume

One obstacle that appears most often in the calculation of improper integrals, called "*ligación a la compacidad* " (Schneider, 1991), which associates a finite area with closed delimited figures. This obstacle seems to be compounded by the lack of coordination between the graphic, algebraic and numerical records, besides the lack of examples and counterexamples in the field of numerical series. In the case of integrals of functions as

1/x and $1/x^2$, some students tend to consider them as the area and volume of a single function. When working in R^2, they have difficulty in differentiating both integral areas under different curves. This is an obstacle linked to the lack of coordination between records, and non flexibility in coordination. In that sense, we consider it´s important to offer students activities which combine different representations. Such is the case with the sequence of graphics which represents 1/x and $1/x^2$ in different intervals: [1,10], [1,1000] and son on, so that students can distinguish the two curves and guess about convergence, as it´s shown in the first and second graphs. The last one is particularly interesting, from a better approximation, considered in the interval [700,1000], giving best guesses about convergence (Figure 1).

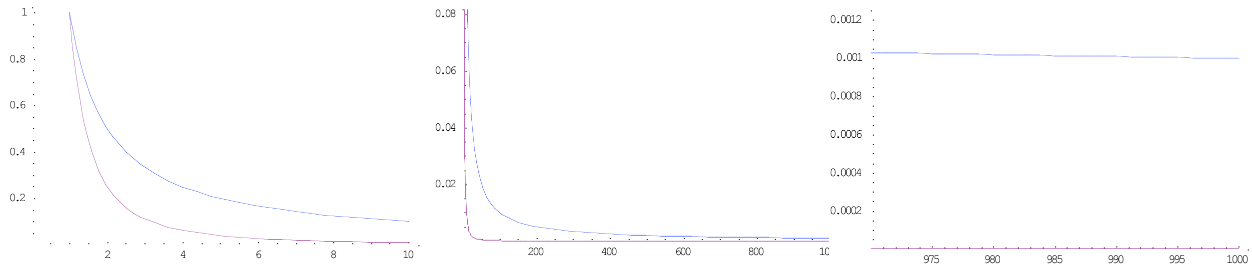
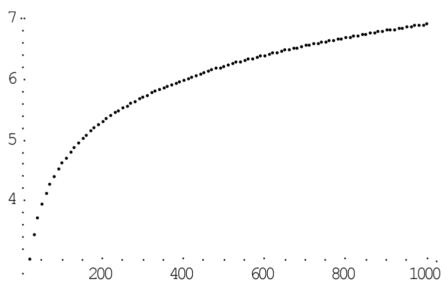


Figure 1. The function 1/x is in blue and $1/x^2$ is in purple.

In addition, the association with the number record allows to corroborate the assumptions about the convergence of each of the curves. The values in Table 1 are: {x, $\int_1^x \frac{1}{t} dt$ } from x = 1 with increments of 50 units, and the graph represents the area under the curve for each value of the area in the table (Figure 2). The combination between both, numeric and graphic records, allow us to infer that the integral doen´t converge.



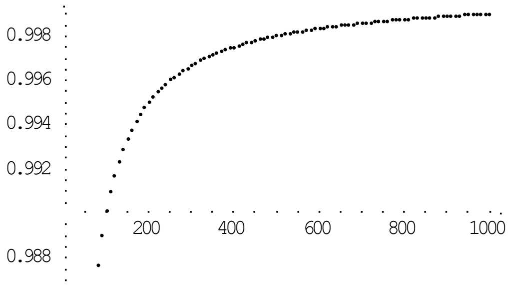
{{1.,0.},{51.,3.93183},{101.,4.61512},{151.,5.01728},{201.,5.3033},{251.,5.52545},{301.,5.70711},{351.,5.86079},{401.,5.99396},{451.,6.11147},{501.,6.21661},{551.,6.31173},{601.,6.39859},{651.,6.47851},{701.,6.55251},{751.,6.62141},{801.,6.68586},{851.,6.74641},{901.,6.80351},{951.,6.85751},{1001.,6.90875}}

Table 1

Figure 2

In the case of $1/x^2$, the values are:

{x, $\int_1^x \frac{1}{t^2} dt$ }, taken with the same frequency (Table 2) and the area under the curve for each value of the area in the table (Figure 3). In this case, the combination between both, numeric and graphic records, allow us to infer that the integral converge. A subsequent algebraic analysis and widespread property for $1/x^n$ can lead to get more general conclusions about the behavior of the hyperbolic functions.

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{{1.,0.},{51.,0.980392},{101.,0.99
0099},{151.,0.993377},{201.,0.9950
25},{251.,0.996016},{301.,0.996678
},{351.,0.997151},{401.,0.997506},
{451.,0.997783},{501.,0.998004},{5
51.,0.998185},{601.,0.998336},{651
.,0.998464},{701.,0.998573},{751.,
0.998668},{801.,0.998752},{851.,0.
998825},{901.,0.99889},{951.,0.998
948},{1001.,0.999901}}
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Table 2

Figure 3

In the same sense, the obstacle of "homogenize dimensions" (Schneider, et. al, 1991) which consists of attributing a volume the properties of the area that generates it, as a solid of revolution, is compounded, among other factors, by the absence of coordination between records. Most of the university literature intends to consider a volume as an overlay surface. Or, through the study of the first two dimensions is possible to find the third, or a volume is an area multiplied by a height. (De Burgo, 2007; Larson, 2006; Stewart, 2007; Zill, 2008).

The most common conceptualization is (Stewart, 2006):

Definition of a solid volume
Consider a solid S which lies between x=a and x = = b. If the area of cross-section S at the plane Px (perpendicular to x-axis), which passes through x and it´s perpendicular to the x-axis, is A(x), where A is a continuous function, then the volume of S is $V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx$

Thus, the volume conceived as consisting of different plates, with the assumption that the area of the "generating" blade (when it comes to a volume of revolution) is infinite, the usual conclusion is that the volume will also. Artigue (1995) describes this obstacle is associated with the implicit and uncontrolled jumps between domain of objects and geometric figures when simultaneously magnitudes of different sizes are handled (consequently the union of magnitudes necessarily correspond to adding measures).

One consequence of this reasoning, is that equal perimeters hold equal areas and equal areas generate equal volumes. An interesting counterexample is to propose the calculation of the area between x and x^2, on the one hand around the x-axis (y = 0), and, on the other hand around y= 2. A proper analysis, through the geometric representation of the problem, allows to understand that the areas match but the volumes are different.

Therefore, as there are figures of infinite perimeter with infinite area, students usually assume that all the figures of infinite perimeter must enclose an area infinite. Schneider (Et. al, 1991) relates, also, this obstacle with the problems caused by the use of the infinite now: if students conceive the volume as an added of flat sections, they will attribute the result of volume, as the limit of added areas and the properties of their elements, the sections, will be transferred to the volume generated.

In this context, students, in first year of university, address the Integral Calculus from the Riemann sum, in which the area is no longer defined as a geometric object, but as the result of a calculation in accordance with a procedure given, where limit and summation take place. No one takes into account the difficulty of linking the area with the process that allows to add infinite amounts infinitely small. Often, from the logical point of view, this reasoning lacks of being able to give a precise meaning to the concept of infinitely small amount. Associated with the foregoing, appears the difficulty of the three magnitudes that are present when we define the Riemann sum: rectangles, segments of which are reduced and the curvilinear area to be determined. The approach, through upper and lower amounts, for example, of rectangles, is built on an existence theorem that ensures that these amounts exist without ever to justify the student, nor his existence, nor their convergence (Tall, 1992).

THE HISTORY OF MATHEMATICS TEACHING AS A DIDACTIC RESOURCE. THE GENETIC METHOD

Introduction

For many reasons, it´s not easy to identify ways to implement The History of Mathematics in education, as it depends, among many other factors, the educational level, the subjects and specific problems, historical knowledge of professor, his interest in interdisciplinary work, his prediposing and ability to perform the transposition didactic, adaptation, reconstruction, recreation and transformation of institutionalized historical knowledge (as useful knowledge) in learning to teach within the historical resources pre-selected as viable in classroom, and, moreover, without falling into anachronistic exhibitions that distor the past in an attempt to describe and interpret it with existing instruments of our notation, language and mathematical terms (Gonzalez, 1992).

Miguel de Guzman (1992) argues that the creative immersion into the difficulties of the past feeds the possibility of extrapolation towards the future. In that sense, Kline (1992) agrees, on the one hand, that the historical perspective gives a more panoramic view of mathematical problems in order to gauge more accurately the importance of various items, which are better articulated within a general context; and on the other, that history can give global perspective on this topic and relate the subjects, not only with each other but also with the central lines of mathematical thinking. In addition, the study of history allows knowing the emergence of epistemological difficulties that present a great similarity with those of students, and therefore facilitates the identification of epistemological obstacles in learning.

There is a repertoire of important issues and problems that can be specially treated following its historical evolution, including the problem of tangents to curves, and the problem of squaring the curve, where you can go back to Archimedes, whose mechanical method points towards the indivisible, while his exhausción method foreshadows the limits of aritmetización Analysis (Gonzalez, 1993). In both problems, tangents and quadratures, there is a slow transition from centuries of mathematical creativity, a suite of brilliant mathematicians are shaping infinitesimal methods and techniques of an enormous heuristic and intuitive value, which call into question the rigor, and bind to ask transcendent epistemological questions about the relationship between invention-discovery processes and presentation-demonstration.

In the view of authors like Gonzalez (et. al, 1992) and Guzman (et. al, 1992) the most direct way to implement the History of Mathematics at the Teaching, is to test on some particular subjects, through the application of genetic method, which tries to rebuild the psychological atmosphere that envelops every creator moment who has represented a quantum leap in the history of mathematics.

The term genetic appears for the first time in Appendix sixth of the book Foundations of Geometry (Hilbert, 1996), where the famous mathematician gives to it a high and heuristic educational value and counterpose to the axiomatic method. The application of genetic method in teaching, presupposes that, for the understanding of a particular concept, the student has to repeat roughly the historical process that has evolved to the current formulation of the concept. Poincaré (1963) describes his philosophy of cultural genetics and F. Klein (1927) developes a genetic argument in his text aimed to the formation of the teaching elementary mathematics candidates from a higher viewpoint. O. Toeplitz, one of the creators of genetic method, which applies in his book The Calculus, a genetic aproach, says : ” *... The genetic method is the surest guide to this gentle rise [in the study of Calculus], which otherwise is not easy to find. Follow the ongoing genetic is the path that has followed the men in their understanding of mathematics, and you will see that humanity has been rising gradually from the simplest to the most complex. Important developments occasional can usually be taken as an indicator of before methodical progress. The teaching methods can benefit greatly from the study of history* "(Toeplitz, 2007: 26).

In his argument against the deductive interpretation, Kline (1978: 48) is based on the historical evidence and agrees unconditionally to genetic method: *"Each person must spend roughly the same experiences for which their ancestors passed if he wants to reach the level of thought that many generations have achieved. [...]. One can not doubt that the difficulties which the great mathematicians found are also the obstacles faced by the students…”*.

In our opinion, not only the difficulties are the same but students must overcome approximately the same manner as mathematicians throughout history did, gradually becoming familiar with new problems, starting by an intuitive level, that is gradually incorporating methods, techniques, ideas and concepts. Clearly, this repetition of the historical process should not be understood at face value. In the construction of Science, often tortuous way, roads that are sometimes reversed, are walked, so that the current educational development of science can not be linear. Without hiding the student the gradually winding of scientific creation, we must guidethem in the learning process. The application of genetic method in the teaching-learning team performs a reconstruction of history that allows to find the key questions that generate ideas and to meet the needs that led to the introduction of a new concept in a particular historic moment, as well as the inherent difficulties related to the delivery of some ideas and the solving of some problems, difficulties, which, as noted Kline (et. al, 1978), are manifested, also, in learning the same concepts and in solving the same problems.

Possible parallelism between the historical development of calculus and the cognitive development of mathematical ideas

The story allows us to understand the difficulties of our students in understanding the concepts of limits, continuity, derivative and integral. Here the historical period of difficulties is very broad, covering virtually since the birth of the rational Mathematics until the end of the nineteenth century, that is to say: from attempts to hide the infinity process in mathematics, conducted by the Greeks, by Archimedes and the exhaución method, up to the reformulation of the new rigorous analysis, undertaken in the nineteenth century by Cauchy, Weierstrass, and other mathematicians like Dedekind; going, as intermediate steps, by the reflections of the medieval Scholastica on the infinite and continuous, which led to the emergence, during the seventeenth century, of infinitesimal methods, which consolidated and generalized by Newton and Leibniz, led to the discovery of infinitesimal calculus.

The obstacle of the "ligación a la compacidad" in students, has its parallel in the calculation of integrals in infinite intervals. The problem was dealt in sigloXVII, when the approach had a more geometric character. The first to address the improper integral, explicitly, was Grégoire de Saint-Vincent, who got his results before Fermat. His motivation was purely to investigate, find and generalize results. He pretended to calculate the volume generated by the rotation of an infinite area, results showed by Guldin, in a surreptitious way, and formalises Tacquet, a disciple of Saint Vincent. (Boyer, 1949).

It is worth noting that in all known cases of the seventeenth century, the interest lied on the potential functions (integrates of the form $\int x^n dx$), except when considering the logarithm function (introduced in the geometric scene in the work of Saint Vincent and Beaune) (Boyer et. Al, 1949). This is where we might find interesting questions about convergence.

As for the problem of "homogeneización de dimensiones" (homogenization of dimensions), it is possible to draw a parallel with the work of Cavalieri, mainly his Geometria indivisibilibus, written and printed in stages between 1620 and 1635, where reflections on the generation of geometric shapes are exhibited. There, the cylinder is generated by a parallelogram and the cone by a triangle, while the surface area of the cylinder is double that of the cone, and the volume is three times. Cavalieri regarded a plane figure as composed of all its lines and a solid as composed by an undetermined number of parallel planes fragments. To avoid adding indivisible, he, instead, determined proportions or relationships. Here, we can see another possible parallelism between the historical development and the cognitive development of mathematical ideas.

This analysis tends to prove that the perception of surfaces (respectively, volumes) as stacking segments (respectively, surfaces) similar to that developed by Cavalieri and others mathematicians in the seventeenth century, although it´s not taught explicitly, it´s present in the mental representations and informal mathematical literacy of students of today. Schneider explains that this fact can explain some frequent and persistent errors in calculating areas and volumes, as well as some difficulties in understanding the modern process of integration. (Artigue et. al, 1995).

CONCLUSIONS

A theory of cognitive development of mathematical thought in the individual, from elementary beginnings through to formal abstractions, requires a cognitive understanding of the formal abstractions themselves.

On the other hand, the relationships between various different representations of a concept, including verbal, procedural, symbolic, numeric and graphic, is necessary to understand it. Empirical evidence traditionally suggests that it is necessary to become familiar with a process before encapsulating it as an object. The computer, used as a didactic tool, is capable of carrying out routine processes, such as drawing graphs, solving numerical tables, which now give the possibility of new learning strategies in

which the objects produced by the computer are the focus of attention before the internal algorithms are studied.

In taking students through the transition to advanced mathematical thinking we should realize that the formalizing and systematizing of the mathematics is the final stage of mathematical thinking, not the total activity. Accordingly to Skemp (), some researchers and teachers try to present mathematics as a logical development. This approach is laudable in that it aims to show that mathematics is sensible and not arbitrary, but it is mistaken in two ways:

First it confuses the logical and the psychological approaches. The main purpose of a logical approach is to convince doubters; that of a psychological one is to bring about understanding.

Second, it gives only the end-product of mathematical discovery (‘this is it, all you have to do is learn it’), and fails to bring about in the learner those processes by which mathematical discoveries were made. It teaches mathematical thought, not mathematical thinking.

In like manner, at the advanced level, teaching definitions and theorems only in a logical development teaches the product of advanced mathematical thought, not the process of advanced mathematical thinking.

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THE APPROPRIATION OF THE NEW MATH ON THE TECHNICAL FEDERAL SCHOOL OF PARANA IN 1960 AND 1970 DECADES

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The subject of this text is the appropriation of the New Math on the Technical Federal School of Parana in 1960's and 1970's. From a historical perspective, founded by Certeau (1982), Chartier (1990) and Julia (2001), the study composed its sources from scholar documents, located on ETFPR files. The study concludes that the ETFPR did not prioritized in its Course Plans, the teaching of the New Math. In this period, the scholar culture of ETFPR was marked by teacher initiatives directed to the elaboration of didactic material suited to the technical courses which were, in that moment, engaged in approaching the scholar mathematics to the technical culture, transforming it in a useful tool for the urgent need of forming the necessary work force to the industrial and technological development of the country.

Since 1960, the international New Math Movement (NMM) has penetrated in several countries schools, seeking to introduce a new language into the scholar Mathematics as well as trying to adjust it to the new challenges brought by the scientific and technological development that emerged in this period.

In Brazil, the movement has increased its force through actions of countless math teachers, like the ones triggered by the Group of Study of Mathematics Teaching (GEEM). The GEEM was created in São Paulo – Brazil and coordinated by teacher Osvaldo Sangiorgi, one of the most enthusiasts members of the NMM in Brazil.

In Brazilian educational context, the technical industrial teaching had a fundamental role in society economic projects, essentially in 1960 and 1970 decades. At that time, the increasing of education levels, especially for poor people, had the main objective of preparing the taskforce for industries, as well as absorbing imported technologies from rich countries. The Federal Technical School of Paraná (ETFPR) [1] carry out a main role, at that moment, of forming taskforce to technologic and industrial development in Paraná State.

Considering the importance of local studies for understanding the national history of the NMM, recognized as a major reformation applied to Scholar Mathematics in a World level basis, the present study aims to understand how the New Math was appropriated by the ETFPR, in 1960 and 1970 decades. According to Valente (2008, p.665):

> The NMM constitutes a fundamental reference to the Mathematics Education as a Research Field. The associated historical moment had triggered the organization and the systematization of scientific activities related to the

teaching and learning of Mathematics. In other words: The NMM made the emerging of the Mathematics Education Research Field.

Oriented by a cultural and historical perspective, the study uses as sources the theoretical-methodological approaches of Certeau (1982), which conceives history as an "operation" that requires for its writing, as a practice activity, of a scientific approach. Besides, Certeau uses the concept of "Appropriation", from Chartier (1990), with the objective to understand the use that scholar agents have made of the New Math, disseminated by the Movement in a scholar culture (Julia, 2001). The study arise questions about changes occurred in the Mathematics discipline offered by the ETFPR, in the NMM discussion period.

The study sources were based in files archived in the Nucleous of Historical Documents (NUDHI) and the General Files of Federal Technological University of Paraná State (UTFPR), in Brazil. In those files, some documents were consulted, such as: Professors Council Proceedings, Class Diaries, Courses Plans, Curricular Grades, Math Books and normative documents.

To confront the date related to the NMM reception, in the scholar practices of the investigated institution, some interviews were conducted with three teachers and an ex-student, which were witness of the teaching, and learning process that took place at ETFPR in 1960 and 1970 decades.

THE PROFESSIONAL TEACHING IN BRAZIL

Professional teaching, in Brazil, has begun in the Imperial time when the first nucleous of professional formation were founded, in Jesuitical colleges and residences. They were called "factory-schools of artisans and other professions" (Manfredi, 2002, p.68). In that period, the most part of manual and manufacturing jobs were done by Slaves. In first Republic, when Brazil was entering a new stage in terms of economical and social development, the professional schools gained a new role, becoming truly technical schools networks. The teaching system of those schools then takes the objective of teaching popular conditions in great Cities. This type of schools, at that time, were directed essentially to poor people, and considered by that fact as a second category school. There was also a great problem of scholar evasion. The most part of the professions that were offered were manual or artisan originated, like joiner, shoemaking and tailor's workshop.

After the 1930 revolution, with the large scale industrial development model adopted by the president Getúlio Vargas, that superseded the agro-exportation model, the factory-schools of artisans and other professions, which were initially the responsibility of agriculture ministry, became part of the new created Education and Health ministry.

In the New State Period, the professional education has the same role of the previous period, which was directed to the poor classes. On the other hand, the secondary course was directed to elite classes. This duality was strongly discussed in the

“Pioneers Manifest”, in 1932, which makes the proposal of the organization of academic and professional courses in the same institution as well as the adaptation of schools to regional interests. In spite of that, only in 1942 the pioneers concerns were accepted by Gustavo Capanema Minister, who’s Organic Laws, among other things, rebuild the Industrial Teaching. According to Cunha (1977, p.55), one of the main factors of the new organization was the Second World War economical context. According to the author, the countries that were involved with the war drastically decreased the exportation of manufactured products to Brazil. One great change proposed by the Organic Laws was the definition of the Industrial Teaching as a secondary course, destined to professional preparation of workers to the industry. With that, the industrial courses students could enter superior courses related to the corresponding professional course.

In the same period, complementary legislation in professional teaching, the edict-law 4.048 of 22nd of January, 1942, created a professional teaching system which was “parallel” to the official system, sustained by enterprises. This new system, nominated National Service of Industrial Learning (SENAI), was supported by the Industrial Confederation and had the finality of organizing and administrating the Industrial Learning Schools of SENAI all over the country. The motivation being the creation of SENAI was that, due to the extinction of the “factory-schools of artisans and other professions”, the old tasks of those schools then became an obligation of the Industries. So, professional enterprises assumed the task of preparing their own taskforce through SENAI and became, gradually, the inspiring model to the technical education for Brazil in later years.

Organized in two cycles (gymnasium and collegial), the first, brought by the Industrial Schools and second, by the Technical Schools, and systematized through the Organic Laws, technical education remained as a branch of education leading to the formation of professional demanded by the production system, therefore, a terminal branch of education. In the 1950’s, through the 1821 Act, the forming students from technical, industrial, commercial and agricultural secondary courses were able to access university courses, provided if they submit to the demands of college entrance examination.

At the end of 1950, with the new National order “education for development”, occurred, during the administration of Juscelino Kubitschek, the reform of Industrial Education. With the Law 3552/59, federal technical schools have been given own legal personality, introducing administrative, educational, technical and financial autonomy and leaving them to constitute a uniform system, with organization and similar courses.

According to Cunha (1977, p.81), despite the autonomy given to technical schools, the control was taken by the Ministry of Education. This control was even increased by the Direction of Industrial Education (DEI) fixing the minimum required curriculum for technicians certificates in specific areas. Among other functions, DEI was responsible for development of curriculum guidelines, the evaluation system, examinations and promotions, besides the development of teaching materials, courses plans and school performance indicators.

At that time of growth and improvement of the Brazilian industrial chain, the spirit of the technique has been widely sown in industrial schools throughout the country. The work of the technical, according to Cunha (1977, p. 30), "begins to depend more on their knowledge than their manual skill or ability of direction"

With the Law of Guidelines and Bases of Education (LDB), which restructured the education in three Degrees: primary, middle and high, technical education began to be offered in three ways: industrial, agricultural and commercial. It was only with this Law that in fact the entry to high education was consolidated for students of professional education.

From 1960, more and more young people were seeking high education as a mean of social ascension, as the economic model of concentration, income left no other alternatives. According to Cunha (1977), in that decade, the social-economic profile of students in technical courses was changing. The number of technicians enrolled in high education during the period between 1962-1966 (about 33%), showed that students of the technical industrial courses hoped that the function of the courses were propaedeutic, an instrument of social ascent.

THE MATHEMATICS DISCIPLINE IN ETFPR, AT NMM PERIOD

According to the Information Bulletin of the Brazilian-American Commission of Industrial Education (CBAI, 1960e, p. 4) [2], the qualified professional is:

> [...] the professional who knows the technology, the practice and still has sufficient basis for progressing into the professional field [...] needs of the concepts of general education as math, drawing, as well as extensive knowledge of technology related to their profession for the development of new techniques and improving of his work.

Considering Mathematics as a basic discipline for the technique culture of students, the biggest challenge that was presented to the teachers of technical courses was to contextualize the content, from problems of practical applications in technological world.

According Clemente (1948, p. 86):

> [...] it is usual to say that mathematics teaches reason and, in industrial education, this proposition assumes a broader character. It's the Math that plays the most important role in the mental training of specialists. Therefore, follows that the teacher of mathematics has, perhaps, the most important part in the sum of knowledge that will form the expert Professional.

In this article, Arlindo Clemente proposes that the teacher of mathematics workshop must bring the factory into the classroom and seek to solve real problems of the job, replacing abstract mathematical problems by concrete ones.

> The mathematical reasoning is the element that will transform the older worker, empirically formed, in the modern workman much more capable, with a greater intellectual capacity. And, no doubt, this parcel of culture is one that will give the worker the possibility of connecting his brain to his hands. This is the function of mathematics in the education industry. (Clemente, 1948, p. 87)

The main concern of Clemente was the practical application of mathematical concepts to technical disciplines of industrial education and the choice of essential and minimum contents, necessary for the training of technicians.

The article by Martignoni (1951, p.695), "The Mathematics in Practice and Education," published in the Bulletin of CBAI, in July 1951, also highlights the importance of mathematics to bring the workshops and cut the superfluous. His speech is full of pragmatism, questioned the need to study the contents that are not directly related to the practical application. Admit that the science math is the reason for scientific progress, but more elaborate math that should be left for further studies because it will not meet the purposes of technical courses under the guidance of CBAI, the Math should have a strong character practical and utility. Meanwhile, the Federal Technical School of Parana, already in the late 50, faced major problems with teachers of the Industrial Technical Education, focusing on the quality of courses. Then Director of Technical School of Curitiba, Dr. Lauro Wilhelm, indicated as early as 1959, two major factors for the low quality of technical courses: the poor training of teachers of general education and technical culture and lack of control over the teacher' activities.

In the late of 1950, the discussion on the mathematics in industrial technical courses had national repercussions. In III Brazilian Congress of Mathematics Education (Ministério da Educação e Cultura, 1959), held in Rio de Janeiro in 1959, coordinated by the Campaign for Improvement of Secondary Education and Broadcasting (CADES), the Industrial Education, whose committee was directed by Arlindo Clemente who presented for discussion, a Program dedicated to the teaching of mathematics in technical courses, highlighting the math in the workshops and the correlation of mathematics disciplines culture technique (Ministério da Educação e Cultura, 1959, p. 28).

NEW MATH TRACES OF ETFPR

The modernization of Mathematics was associated with betting on technical progress. For Valente (2006, p. 39), "the Math was valued as part of a scientific training that would have continuity in Higher Education and to do so was needed rapprochement between the approaches of mathematics in Higher Education and in secondary, in conceptual terms, methodological and language". This approach to the mathematics of Higher Education expressed on the main features in NMM: The accuracy, precision of language, deductive method, and generally higher level of abstraction, use of contemporary vocabulary, thought axiomatic among others.

However, even taking the Technical School teachers participated in the preparation of textbooks of New Math of the group's Center for Research and Dissemination in Mathematics Teaching (NEDEM) in the Parana State College (CEP), these actions do not seem to result in upgrade programs of Mathematics. So they show the "Daily Class" (document 6) [3] the years of 1967 and 1972, teachers of the ETFPR Industrial Gymnasium, which do not show any trace of New Math.

In oral testimony, the teacher E1 [4] reported that the books of mathematics, used in industrial Gymnasium, at end of the 1960's, were Marcondes (1969). The collection was divided into three volumes: algebra, arithmetic and geometry. Referring to the edition of 1969, found the lack of content of New Math.

It is important to remember that some Mathematics teachers, employed by ETFPR in the second half of 1960's, were still students in the Course of Mathematics at the Federal University of Parana (UFPR), and had no authority to his colleagues to propose changes in programs and in the textbooks adopted. Much of these new teachers were in contact with the new contents of Modern Mathematics, run programs developed by teachers responsible by the subject, since their independence was conditioned by teachers who were at school for a long time, a school culture specific for technical school.

Also, at the beginning of 1970's, new teachers of Mathematics were minority. This is confirmed by the testimony of a former student from Industrial Gymnasium: They had some new teachers, but a proportion of 70% were the most experienced teachers (E3).

The teacher E1, in testimony to the researcher, reported that the first time he heard Theory of Sets was in 1967, when his teacher asked him the option to work on this topic. In 1970, when he graduated in Mathematics, by UFPR, began working in the State Network for Teaching and ETFPR, teaching Mathematics belonging to the gymnasium course. According E1, the network state of education first adopted the Mathematics book of NEDEM and then Oswaldo Sangiorgi´s book. He said he came to work a full year in the State Network with Theory of Sets. Already in ETFPR were taught some notions of collections, but that was not exagerated (E1).

In 1966, theacher Ricardo took over the direction of ETFPR. The entry of this new director would give new direction to the organization of teaching-learning school. He took in baggage, more than to the experience of CBAI, the coexistence with the Americans and the commitment with the institution and students. The strong American influence received by the new director would be largely responsible for the ideas of method, rationality, profficiency that would come with greater intensity, being part of school culture of ETFPR. In his testimony, Professor Ricardo Luis Knesebeck reported that first, as coordinator of instruction, and then director, implanted, demanded, draconian by the program of education for all teachers, it was an something absurd to teach and don't commit with anything.

The document "Content to be determined" (document 11) [6], prepared by teachers of mathematics and approved by the Coordination Didactic, in 1969, showed that the program was based on the sequence of the contents in the collection, books in Quintella (1966), which until 1970 did not have any trace of New Math, and Theory of Sets, relations, matrices, etc. Odds specified in the "Pilot Program" (document 12) [7] published by GEEM, in the year 1968.

In oral testimony E1 the teacher said that teachers closely follow the book, the first to the last page and the Head of the Department selected by the exercises in the book that the teacher should do. In his opinion, this hand method works very well. In the Mathematics Program in 1st years (document 11), we found a topic: "General Review of the 1st cycle of matter." This may be an indication of the teachers concern to maintain a certain quality of education that could be harmed, because of the low quality in Mathematics taught in the gymnasium.

In the analysis of the goals of textbooks delivered to students, called "Auroras", observed that in the year 1973, compared with the program in 1969, the complex numbers and trigonometric equations were removed, and simplified the study of vectors and orthogonal views. We also note that a greater emphasis was given to the trigonometric functions.

In the “Auroras” program in 1975 some contents were evaluated:

> I - SET - Goal 1: Operating with sets. 1.1 - Determine the union of sets. 1.2 - Determine the intersection of sets. 1.3 - Determining the difference between two sets . 1.4 - Determine the complement of a set. 1.5 - Correctly use the symbols of the theory of sets. II - NUMBERS (NUMERICAL SETS BASIC) - Goal 2 - Understand the fundamental numerical sets (...). III - RELATIONS AND FUNCTIONS. Goal 3 - Represent graphically relationship and function. (...)3.3 - Determine the Cartesian product between two sets. (...)

This portion of the manual for evaluation of the student confirms the evidence E1 on the introduction of theory of sets for the students of secondary technical course and the new approach to the concept of function according to modern mathematics. the notion of variation and functional dependence of the functions was virtually forgotten over the NMM that adopts the design of structural function of Bourbaki.

In the year 1975, the term "field of existence" has been replaced by "dominion" and "image" of trigonometric functions, the term used in the book Iezzi et al. (1973) [8]. Making a comparison between the "Pilot Program" (document 12), prepared by GEEM in 1968, for the first two years of secondary education, noted that the ETFPR's program, although more extensive, included topics such as the trigonometric functions and resolution of triangles, suggested by the group of São Paulo.

In 1975, an ETFPR is a complete revision of the programs of Algebra (Math I). With adoption book Iezzi´s et al., (1980), the topics turn to a deal with sets, sets numerical key, full study of the functions of the 1st and 2nd grade, depending Exponential, logarithmic function, the study inequalities of 1st and 2nd grades, exponential and logarithmic. The subjects are addressed in accordance with the "Pilot Program" (document 12) suggested by GEEM in 1968.

Iezzi's book presented the contents of duty by a graphical approach. Separating each chapter, there was an example of mathematics application in today's world. There was a concern with the formal mathematics, but not so exaggerated. Indicated at the end of the book, had several references about the Modern Mathematics.

We can noticed that probably the book's Iezzi *et al*, (1980) deals with the theory of sets to meet a market need, as a warning Kline (1976, p.135) "Other texts begins with a chapter on the theory of sets, It was then back to the traditional math and would henceforth no longer refer to the theory of sets or any other topic in modern mathematics".

The book's Gelson Iezzi *et al* have consolidated a discussion of teachers curriculum modernization, already presents among the Mathematics of ETFPR. In his testimony, teacher E1 said that he and his colleagues in the early of 1970 began to define functions as a particular case of the relationship between two sets (a structural design adopted by the NMM) rather than as a functional dependence as was discussed of Ary Quintella´s book. According E2, a teacher of ETFPR the 1960's, the technical course did not give much emphasis to the theory of sets, it was an education more focused on practice. One possible explanation for the slow integration of the Modern Mathematics in ETFPR could be one of the goals for Educational System in the ETFPR "(document 4) [9] as defined in 1972:" Cut programs of study fictitions topics". Would be "fictitious subject", the content broadcast by the NMM? Would be inappropriate to technical education?

In the first half of 1970's, despite the strong tendency to follow faithfully the textbook, some mathematics' teachers of ETFPR started developing their own material to work with students, such as "Geometry of Space Material" (document 15) [10].

The exercises in these first worksheets have not any relation with the technical matters because there was a culture of integration between these areas. According to the interviewee, the teaching of mathematics was not aimed at career academies: No, it was generic. At the time, from 1969 until 1974, it didn't have a very great integration between the teachers of general education and culture specific; they worked half apart (E3).

In 1970's, with the support and encouragement of the Department Mathematics Coordinator, the production of teaching material itself was improved and marked in a more intensive way the culture of ETFPR. This initiative was not alone, it was occurring in several federal technical schools in Brazil. In ETFPR, this initiative was consolidated in 1980's and resulted in a collection mathematics books directed to the Technical Education.

FINAL CONSIDERATIONS

The study indicates, some aspects emphasized by NMM, as the theory of sets, the axiomatization, the new mathematical language, laden with symbolism, seemed incompatible with the needs of the students training who wanted a technical school in the 1960's and 1970's.

Concerned to offer a "practical education", required by technical training, an ETFPR not prioritized the teaching of modern mathematics in their courses, at the top of the movement. The testimonies show that there was an insertion of non-official "some" ideas of NMM and this can be evidenced by the few traces of Modern Math, documents found in the school.

The study shows that only from 1970, some contents of New Math were introduced in the course of school, and that means textbook from 1980, Mathematics teachers ETFPR started of the preparation of a Mathematics textbook colletion, putting an old idea to feature a "practice" to discipline by proposing a specific methodology able to articulate the rationale, graphic interpretations, problems applying physics problems and technical subjects. The weak presence of New Math in ETFPR, far from setting itself as a resistance from teachers, to the ideals of the movement, indicates that in decades in question, a ETFPR wanted to amalgamate a difference in their school culture, slowly making a "creative consumption" of textbooks, strong responsible for the insertion of New Math in Brazilian schools.

NOTES

1. Today is called Federal Technological University of Parana (UTFPR). Use the name Federal Technical School of Parana (ETFPR), like this named because most of the period defined in the study, namely the 1960's and 1970's.

2. Bulletin of CBAI. Brazilian-American Commission of Industrial Education. Educational program of cooperation maintained by the governments of Brazil and the United States Research and Training of Teachers. Vol. XIV, n.5, 1960e, 16p.

3. Document 6: Diaries of the course belonging class of 1967 and 1972. - Archive of General UTFPR. 1967 to 1972.

4. The name of the interviewees E1, E2 and E3 was not revealed at their request.

5. KNESEBECK, Ricardo Luis ex-student, ex-teacher of physics, ex-director of the Federal Technical School of Parana. (Interview granted to Gilson Leandro San Mateo - NUDHI / UTFPR. Curitiba, 16/17 May 1995).

6. Document 11: Content to be established in 1969, 1969, 17p.

7. Document 12: Pilot Program for the school course prepared by GEEM in 1968. Sao Paulo: GEEM, 1968, 5p.

8. The first edition of this book is the year of 1973. In this study found was the eighth edition, published in 1980.

9. Document 4: The educational system of the Federal Technical School of Parana produced by the Education Department through the coordination of the Didactic ETFPR.

10. Document 15: Geometry of Space Materialmade by teachers of ETFPR. Library of UTFPR, s / date.

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HISTORY, HERITAGE, AND THE UK MATHEMATICS CLASSROOM

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Since 1989 the UK mathematics curriculum has been dominated by a culture of testing 'core skills'. From September 2008, a new curriculum places the history of mathematics as one of its "Key Concepts' which is now a statutory right[1] *for all pupils. While the curriculum has changed, there has been virtually no relevant training for teachers, and while the testing regime remains in place, there seems little chance that pupils will obtain their entitlement. This paper examines the problem of teachers' scant knowledge of history of mathematics and proposes a new approach to introducing relevant materials together with a pedagogy which capitalises on recent research, to introduce the heritage of mathematics into our curriculum.*

THE NEW ENGLISH CURRICULUM

The first chapter of Fauvel and van Maanen (2000) considered the political context of the history of mathematics in school curricula. At that time, the UK curriculum[2] was undergoing radical changes, which produced a curriculum based on 'core skills' with modularised[3] lessons that enshrined traditional beliefs about 'levels' of knowledge that produced a 'topic-based' curriculum as a collection of disparate activities rarely connected in any sensible way. Textbook design followed the syllabus, and past test papers became *de facto* part of the curriculum, setting the norms for the new culture, and the emphasis on utilitarianism and examination results produced little serious engagement with substantial mathematical thinking[4]. The latest Inspectors' report on our secondary schools shows that too many pupils are taught formulas that they do not understand, and cannot apply:

> The fundamental issue for teachers is how better to develop pupils' mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently. (Ofsted, 2008, p. 5)

Today, in contrast, a new mathematics curriculum states that for the 11 - 16 age group, "Recognising the rich historical and cultural roots of mathematics" is one of its "Key Concepts" (QCA 2007)[5].

For the last fifteen years very few secondary school teachers have had the chance to discover the range and potential of the contributions that the history of mathematics could make to pupils' learning, and with the pressures of 'teaching to the test' it seems doubtful whether history of mathematics will make any impression in our classrooms while the examination structure remains the same[6]. So, what would 'recognising the rich historical and cultural roots of mathematics' mean in practical terms for our teachers?

In a recent issue of *Educational Studies in Mathematics*, colleagues have reviewed the evidence, both theoretical and practical, and renewed their call for the history of mathematics to be taken seriously as an essential part of the mathematics curriculum. Radford et. al. (2007) argue that an important sense of meaning lies within the cultural-epistemic conception of the history of mathematics:

> The very possibility of learning rests on our capability of immersing ourselves –in idiosyncratic, critical and reflective ways– in the conceptual historical riches deposited in, and continuously modified by, social practices. … Classroom emergent knowledge is rather something encompassed by the Gadamerian link between past and present. *And it is precisely here, in the unravelling and understanding of this link, which is the topos or place of Meaning, that the history of mathematics has much to offer to mathematics education*. (p. 108) (italics mine)

In the terms described above, history stands in opposition to the utilitarian demands of the old curriculum, but having put history of mathematics into the curriculum, the government organization, QCA[7] have now revealed the pressing problems of resources and training. Changes need to happen not only in the classroom but also, and more importantly, in teacher training.

> So, *how can we provide material from the history of mathematics that can be integrated in a meaningful and effective way into the everyday activities of the classroom*?

NOT HISTORY BUT HERITAGE

Ivor Grattan-Guiness (2004) has made an important distinction between the History and the *Heritage* of mathematics. History focuses on the detail, cultural context, negative influences, anomalies, and so on, in order to provide evidence, so far as we are able to tell, of what happened and how it happened. *Heritage,* on the other hand, address the question "how did we get here?" where previous ideas are seen in terms of contemporary explanations and similarities are sought.

> The distinction between the history and the heritage of [an idea] clearly involves its relation to its prehistory and its posthistory. The historian may well try to spot the historical *foresight* - or maybe lack of foresight - of his historical figures, …. By contrast, the inheritor may seek historical *perspective* and hindsight about the ways notions actually seemed to have developed. (p. 168)

> …heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows foundations are dug down, and not necessarily into firm territory. (p. 171)

The interpretation of Euclid's work as 'geometrical algebra' has since shown to be quite misguided[8] as *history,* but as *heritage* is quite legitimate because it is the form in which some of the Arabs modified the Elements when they were creating algebra.

We have to be careful. Deterministically constructed heritage conveys the impression that the progress of ideas shows mathematics simply as a cumulative discipline. In some sense that may be true, mathematics does build upon past achievements, but while we may make stories about the links between the mathematics of the past to the present, the mathematics of the past is not the same as the mathematics of now. As Mathematics Educators we have a means of passing on the Heritage by bringing to the attention of teachers and students the links between the content we find in the curriculum, and hence, what we know of the history of mathematics. In this way it becomes possible to describe significant landmarks in the history of mathematics in terms that teachers and pupils can understand without making impossible demands on their historical capability or on curriculum teaching time.

PROJECT AIMS AND OUTLINE

This project is just beginning. It arises from the experiences of myself and other colleagues in presenting ‘episodes’ from the history of mathematics in workshop form, so that interesting and worthwhile problems arise from the historical context.

At this moment, I am principally interested in providing secondary teachers with professional development materials that start from the important fundamental ideas in the curriculum they have to teach, and to open up the possibilities of developing the concepts involved by finding ‘historical antecedents’ to support the connections between and motivations for these ideas and the possible links between them.

Exactly what form this material may take is still under consideration[9]. The general idea is to produce a series of ‘concept maps’ that are intended to provide a topographical view of the significant features of a particular mathematical landscape[10]. A map can be examined and tackled from ‘inside-out’ and from ‘outside-in’, from following particular trails of thought to obtaining a broader overview of historical development. The ‘unravelling and understanding’ of the links between ideas, is the *topos* that Radford and our colleagues (quoted above) are talking about.

The idea of a map is important here; it is intended to be a guide to how ideas might be connected, not a deterministically constructed list of events. In contrast, most curriculum activities are presented to teachers as a *narrative*, a list of topics to teach in a particular order, and often restricted to some imagined ‘levels of competence’ of the pupils. In many cases, the narrative is also placed in a particular so-called ‘real life’ situation which pupils are quick to spot as unrealisitic.[11]

A map is there for teachers to have the freedom to make their own narrative. They have the responsibility for producing lessons, and it is up to them what parts of the map they want to use, and how they approach the pedagogical problems of dealing with the curriculum in their own classroom. The map can throw light on certain problems, it can suggest different approaches to teaching, it can help to generate didactical questions, but in the end it is there to be used or not, appropriately.The intention here is to develop ways in which the teacher, starting from a particular point in the standard curriculum, will be able to link a conceptual area with important developments in the history of mathematics through the use of ‘idealised’ historical problems and *canonical situations*[12]. There is a deliberate intention not to teach history; instead, - we are dealing with the *Heritage* of mathematics, namely, of the ways and means in which ideas have been interpreted by others at different stages in the past. There is, of course, a considerable literature of historical and pedagogical material to draw on. The practical task is to find appropriate ways in which to link the source material with the curriculum opportunities.

METHODOLOGY AND PEDAGOGICAL APPROACH.

Since the English curriculum now focuses more on what we call the 'process' aspects of learning mathematics, it may now become easier to incorporate the teaching of the 'key concepts' in such a way as to enable the history to emerge from the discussion of *canonical situations* (be they images, texts, or conceptual problems) introduced by the teacher. This approach also has the advantage of being able to link different areas of a standard curriculum, thereby enabling pupils to see connections between parts of mathematics that have been concealed by the traditions of official curriculum organisation. When the text-books and exercises are arranged so that their chapter headings conform to the same organisation as the curriculum, it is most unlikely that pupils will gain any idea that different areas of mathematics are connected at all. In this pedagogical strategy we are concerned with the *dynamics of production of the pupils' ideas* stimulated by episodes from the history of mathematics retold in heritage form. In principle, this is not new. I am advocating the use of a methodology that is already available, which can bring mathematics education and the teaching of history of mathematics together. The principles are well-established, and the use of examples has been a tradition in teaching for many years. However, as Sierpinska (1994) has recognised:

> Pedagogues, of course, think of paradigmatic examples …. of instances that can best explain a rule, or a method, or a concept. The learner is also looking for
>
> such paradigmatic examples as he or she is learning something new. The problem is, however, that before you have a grasp of a whole domain of knowledge you are learning, you are unable to tell a paradigmatic example from a non-paradigmatic one. (pp. 88-89)

This problem is always present in the classroom, but there are many different ways in which we try to alleviate the situation. Grosholz (2005) has demonstrated the role of 'constructive ambiguity' in Galileo's discussion of free fall,[13] and shows that ambiguity can play a constructive part in mathematics since it leads in this case to reading a particular diagram in more than one way. Galileo's argument was put forward in terms of proportions, geometrical figures, numbers and natural language. He was then able to exploit Euclidean results and the arithmetical pattern of the diagram, but in reading the intervals as infinitesimals he led the participants heuristically to his analysis of accelerated motion. Grosholz shows that the use of ambiguity in mathematics occurred not only in the past, but is also present today. Changing the mathematical context by conceptualising new objects and the processes we use to deal with them, changes the ways in which arguments can be understood.

This kind of ambiguity has been shown to provide useful material for classroom discussion. For example, Barbin (2008), has shown how reading a text as a message to an audience can motivate a discussion about the intention and meaning of the author, and consequently of the ways it could be interpreted and understood. Other ways of approaching pupils' learning have been to recognise the learner's inherent abilities and a sensitive focussing on what the pupil can do is fundamental in Gattegno's approach which is the basis of much research:

> The role of the teacher of mathematics is to recognise that a student who can speak has a large number of mental structures which can serve as the basis for awarenesses that will enable him to transform these structures into mathematical ones. (p. 70)

We of course recognise that there is no sure way of posing problems or offering examples, but once done, then the learner's response has to be respected and managed carefully. We have become used to the principles of heuristic teaching, but Brent Davis claims that heuristic listening is also important:

> *Heurisitic Listening ……* is more negotiatory, engaging, messy, involving the heare and the heard in a shared project [which] is an imaginative participation in the formation and transformation of experience through an ongoing interpretation of the taken-for-granted and the prejudices that frame perceptions and actions. (p. 53)

When we engage in mathematical problems we inevitably construct our own examples to help us illustrate the ideas involved, and use these examples as material for personal contemplation or discussion amongst our peers. If we do this as adult mathematicians, why should it be different for pupils? Why is it not possible to develop this idea of self-construction in the classroom?

Since the early 1960s in England, there has been a tradition of producing materials for teachers and pupils that focuses on an individual's learning process and encourages active engagement in, and discussion of mathematical problems[14]. Watson and Mason (1998) and Swan (2006) provide practical guidance in ways of helping teachers to develop pupils' powers of constructing mathematics for themselves in the classroom:

> Our interest is in using mathematical questions as prompts and devices for promoting students in thinking mathematically, and thus becoming better at learning and doing mathematics. … We hope our work will show how *higher order mathematical thinking* can be provoked and promoted as an integral part of teaching and learning school mathematics, through the teacher's leadership and example. (p. 4)

Such publications display through their considerable theoretical analysis and practical experience, ideas for situations that are generic and offer ways for teachers of promoting 'Learner Generated Examples' *applicable at all stages* of teaching and learning mathematics. The materials are prepared to promote the kinds of activities that focus on ambiguity, raise doubts about interpretations, and encourage the learner (and the teacher) to develop a security with mathematical ideas that enables them to engage in intelligent questioning and active discussion of the problems concerned. A number of teachers have engaged in this pedagogy which raises pupils' learning above mere acquisition of skills, and helps the pupils to develop their own cognitive tools and to achieve a higher order of mathematical activity.

THE MATERIALS: DESCRIPTION AND EXAMPLES

Completing the Square is one of the drafts that has been used in a number of classrooms[15] and covers is a traditional area of the curriculum showing some of the connections between the stages to the solution of quadratic equations. It comprises a series of links from one period to another, stressing the transformation of the ideas from simple surveying to 'cut and paste' problems in Mesopotamia, to more sophisticated procedures of 'dissection and re-arrangement' in India, and how the problems were transposed and represented within the more abstract ideals of classical geometry in Greece. The classification of problems and the introduction of algebraic concepts by the Arabs, eventually found their way into Europe and resulted in the attempts to find solutions of different types of equations. The materials provide plenty of opportunities to discuss the development of geometrical and number concepts and the way these were represented in text and diagram form (ratios, proportions, integers, fractions, rationals non-rationals and eventually 'imaginary' numbers). Key ideas like the different forms of representation, appropriate notation, and whether a particular procedure is 'allowed' in a given context, can be discussed, and show how finding representations for 'impossible' numbers like $\sqrt{3}$ or π can have a liberating effect in allowing new ideas to flourish. And, of course, there is the ever-present idea of 'infinity' to be explored. The material has been gathered from the expert analysis of many historical documents[16] and the use of published research to attempt to identify and characterise significant moments in the evolution of particular ideas. From these examples I have taken, not only the translation of the documents into 'modern' language, but something of the pedagogical interpretations, so that these might be brought into the modern classroom and used in creative ways. The material is designed so that it can be used in 'episodes' in the normal course of teaching in school (not necessarily in 'historical' order). Included are notes and references to the historical background, and 'pedagogical notes' aimed to help teachers raise questions and see where the material can be used in their classroom.

In this way, selections can also be used as a basis for teachers' professional development both in the historical and mathematical sense.

There are optional entry (and exit) points to the material that allow considerable flexibility in its use. These characterisations of the 'episodes' are not necessarily unique, neither are they exceptional. They are significant in the sense that they apply to particular topics (or lack of them) in the English mathematics curriculum, and are each recognised as an interpretation of a particular context in our heritage. This is where the historical process can be described *in terms communicable to a modern school audience* and furthermore, the teaching is specifically designed to focus on the pupils mathematical activity *in the contemplation and discussion of the problems*, and their opportunity to engage in a dialogue with the material. It is hoped that together with the development of the pedagogical methodology described above, we may have a chance of truly beginning to recognise "the rich historical and cultural roots of mathematics" in our classrooms.

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UK WEB BASED DOCUMENTS

a) Pedagogical sources:

Changes in Mathematics Teaching Project: http://www.cmtp.co.uk/

Deep Progress in Mathematics

http://atm.org.uk/reviews/books/deepprogressinmathematics.html

b) Government documents:

QCA Website www.qca.org Maths_KS3_PoS.pdf and QCA Maths_KS4_PoS.pdf

http://curriculum.qca.org.uk/key-stages-3-and-4/subjects/mathematics/index.aspx

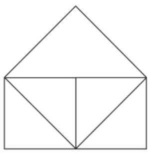
Ofsted UK (September 2008) Mathematics: ***Understanding the Score***

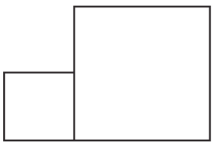
Download review of mathematics led by Sir Peter Williams published June 2008

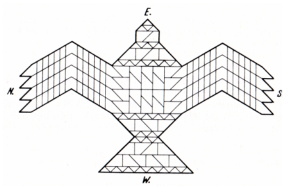
http://www.ofsted.gov.uk/Ofsted-home/Publications-and-research/Documents-by-type/Thematic-reports/Mathematics-understanding-the-score

COMPLETING THE SQUARE (Some Samples)

1. Indian Area Methods.





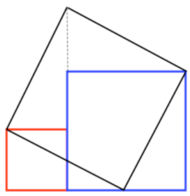


(a) (b) (c)

These diagrams and are inspired by practical Altar Building rules from the Sulbasutras, (15th - 5th Centuries BCE), (c) is the 'Kite Altar' still used in Kerala.

Challenge: It is easy to see how the combined areas of two equal squares can be found (a); with only a rope for measuring and drawing arcs, what about the combined area of (b)? Allow time for experiment and discussion of pupils' procedures. Ask pupils if they can find any more solutions. Does it work for any size of squares?

Activity: Use square dot-lattice paper to draw squares with a dot at each corner and no dots on the edge. Find their areas using the smallest square as the unit. Discuss methods of dissecting the squares to find equivalent areas and how these may be combined. Display diagram (d) and discuss 'transformation of areas'.



(d)

There are a number of variations of diagram (d) possible. Further developments could explore the visual dynamic of diagram with software; extension to rectangles and other shapes; identifying basic properties and justification of procedures.

Clearly, this can link with ideas from Mesopotamian mathematics and Euclid Book II.

2. The Babylonian Algorithm.

A number game: "I am thinking of two numbers, their sum is 7 and their product 12, what are the numbers?" Extend with increasing pairs of sum and product numbers, encourage pupils to discover the original numbers. Pupils to challenge each other, share results, and find a way of writing instructions or developing a notation.

Introduce a standard algorithm: 'Take half of 7, square it, subtract 12 from this square and find the square root of the result, then add and subtract this square root from half of 7.' Use this to test other pairs. If it works for integers, it should work for rationals, so try it with simple fractions. This algorithm originates in Mesopotamian mathematics and variations of it are found in Al-Khowarizmi, Fibonaci and Cardano.

Extensions what happens when the pairs are 7, 11 and 7,13? These simple variations give non-rational ($\sqrt{5}$) and complex results ($\sqrt{-3}$) respectively.

Note 1: I see no problem in introducing quite young pupils to ideas like this. The process of 'following the algorithm' with simple numbers allows pupils to arrive at results which mirror in the discovery of these 'impossible' numbers.

Note 2: In this context, we also have the opportunity of introducing an iterative solution method for finding square roots, linked to the famous Old Babylonian tablet YBC7289. Discussion about the number that when multiplied by itself can produce 2, can lead to pupils' experimenting and developing their own methods of 'trial and error'. This is also one of the important opportunities to contemplate how we can manage and understand an infinite process.

Note 3: Finding a suitable notation is an important part of mathematical history and communication. In most cases in school mathematics notation is given unmotivated to pupils. Situations where pupils are challenged to communicate ideas to their peers through such examples provide opportunities for exploiting historical analogy.

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[1] A 'statutory right' means that *by Law*, *all pupils at primary and secondary level* have the right to be taught about the "rich historical and cultural roots of mathematics".

[2] The UK mathematics curriculum applies to England and Wales. Due to government devolution Scotland and Northern Ireland have different curricula, regulations and examination systems.

[3] Modules purport to be convenient 'packages of knowledge' within the curriculum, with a well defined and limited range of knowledge. They are consequently easy to 'teach' and easy to pass.

[4] There are, of course, a number of exceptional teachers who have overcome these difficulties.

[5] The Key Concepts are: Competence, Creativity, Applications and Implications, Critical Understanding, and the Key Processes are: Representing, Analysing, Interpreting and Evaluating, Communicating and Reflecting. Applied to all pupils from age 11 to 16 (Key Stage 3 to Key Stage 4).

[6] Recently, the government has decided to abandon the tests at KS3 (age 14), but persuading teachers out of their current ‘mind set’ is going to take time.

[7] The Qualifications and Curriculum Authority, the Government sponsored body set up to maintain and develop the national curriculum and associated assessments, tests and examinations.

[8] Typically, this is done with Euclid II,4 and described as ‘completing the square’, but see the examples in Katz (2008)

[9] Today, many options present themselves: texts, posters, PowerPoint, DVD are all possibilities.

[10] I make no claims that such a map is (or even could be) ‘complete’.

[11] A serious indictment of this situation is Paul Dowling’s (1998) work on the sociology of mathematics education.

[12] By a *canonical situation* I mean a diagram, or a way of setting out a problem or process which is developable, has potential to represent more than one idea, and is presented to students to encourage potential links between apparently different areas of mathematics. See the Appendix for an example.

[13] Galileo (1638) Discorsi e Dimostrazioni Matematiche Day 3, Theorem 1, Proposition 1 (Dover edition p.173).

[14] This kind of material was introduced by the Association of Teachers of Mathematics, and has been its enduring hallmark. It is the result of a tradition of collaborative research and writing where texts and other materials have developed a particular type of pedagogical practice by offering examples of classroom work which require discussion, involve heuristic forms of reasoning, analogy and inference, and encourage the learner to create and verify their own examples.

[15] Over the past five years U have used this material, in whole or in part, with various groups of pupils from age 11 to 17, with teachers, and with graduate teacher trainees. I gratefully acknowledge their feedback, which has been most useful.

[16] For example, over the years I have been able to access the specialised work of many researchers on Ancient, Classical, Mediaeval and Renaissance mathematics. Now we can find substantial examples of much of the ancient mathematical material collected and specially written up in (Katz 2007).

INTRODUCTION OF AN HISTORICAL AND ANTHROPOLOGICAL PERSPECTIVE IN MATHEMATICS: AN EXAMPLE IN SECONDARY SCHOOL IN FRANCE

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Abstract: To introduce an anthropological and historical perspective in mathematics from middle school is a challenge that we have tried to face for several years. We first present what we mean with "an anthropological and historical perspective in mathematics", our theoretical references, including didactics one, and our motivations for choosing the thematic of irrationality. In the second part, we will present elements of three experimentations done at grade 8 (13-14) and grade 10 (15-16).

Key-words: History of mathematics – Anthropological approach – Didactics of mathematics – Epistemology - Irrationality

I. MOTIVATIONS

In France, attempts to introduce an historical perspective in mathematics have been developed for several years, in particular, but not only, through the IREM Commission on History and Epistemology of Mathematics[3]. Some historical elements are also often introduced in textbooks (but most often without taking in account mathematical considerations). Beyond this, a crucial issue in a didactic perspective is the way it is possible to articulate historical elements with the mathematical knowledge to teach at the various levels of the curriculum. To approach mathematically historical texts necessitates most often an important effort for their understanding, and the possibility to put in relation these texts with the mathematical contents to teach is difficult and far to be an evident choice, due in particular to the fact that the modern concepts are more efficient to solve the related problems. This could explain the rather common choice of limiting the introduction of history to informative aspects aiming mostly to motivate the students. Although this aspect is not to neglect, because it could permit to modify the common representation of mathematics as timeless knowledge, it does not take in account the potential contribution of History of Mathematics for the learning of Mathematics itself. As Bkouche (2000), we consider that an historical perspective in teaching of sciences « can be inserted less as a motivation than a *problematisation* » in the following meaning: "Epistemology of problems aims to analyse how the problems that lead humanity to elaborate this mode of knowledge that we name scientific knowledge have modelled the theories invented in order to solve these problems"[4].

II. THEORETICAL BACKGROUND

II.1. Anthropological fundaments of mathematics

In continuity with Tardy (1997), we have chosen to situate the historical perspective in the field of Anthropology. Chevallard (1991) considers that Didactics of mathematics is the “advance point of the anthropological continent in the mathematics universe”, that specifies its place in the field of Anthropology. In this perspective, he mainly studies the didactic transposition, i.e. the transformation underwent by mathematical knowledge when they are taught and used. For him, “present epistemology” studies the question of knowledge production while he considers Epistemology in the broader sense of Anthropology of knowledge.

In this paper, we refer to the sense of “present epistemology”, including anthropological considerations, according to Kilani (1992) that Anthropology search relations between local knowledge or specifics discourses on cultures to global knowledge or general discourse on humanity.

II.2. Genetic psychology and Anthropology

Genetic Psychology elaborated by Piaget questions Anthropology. Opposite to Piaget, present Anthropology does not consider hierarchy among different stages. The stages that Piaget has distinguished (practical intelligence; subjective, egocentric, symbolic or operative thought) cross the interrogations of Anthropology on the relationship between culture and thought, leading to debate around myth and rationality, magic and science and the way to pass from an aspect to another. Anthropology states that operative and symbolic thoughts have different purposes; that they do not exclude each other, coexisting in a singular person as well as in a given society[5]. Moreover, it could be thought that Imaginariness as well as reason could play a role in scientific discoveries (Kilani, 1992)

Following Vergnaud, we can add that in mathematics activity, these different modes of thoughts are necessary and complementary.

> “Explicit concepts and theorems only form the visible part of the iceberg of conceptualisation: without the hidden part formed by operative invariants, this visible part would be nothing. Reciprocally, we are unable to talk about operative invariant integrated in Schemas without the categories of explicit knowledge: propositions, propositional functions, objects, arguments.” (Vergnaud, 1991, p.145)[6]

II.3. The epistemological model of « milieu » in the Theory of Didactical Situations (Bloch, 2002)

- ✓ **About the concept of milieu**

The concept of « milieu » plays an important role in the Theory of Didactical Situations (Brousseau, 1997). Several authors have reworked and developed this concept, which was one of the thematic of The 11th Didactic Summer School in

France in 2001. From our perspective, the models of *milieu* presented in this frame by Bloch is particularly enlightening. In her course's introduction[7], Bloch indicates:

> "In this course, we aim to attempt a clarification of some fundamental concepts of Theory of Didactical Situations, and for this purpose to propose a reorganisation of the models of milieu of this theory to predict and analyse teaching phenomena. It is clearly an elaboration aiming to classify the theoretical elements related to the *milieu* according with their functionality (from knowledge; from experiment; from contingency)" (Bloch, 2002, p.2)[8].

This leads her to propose the three following models: the *epistemological milieu* that concerns the cultural knowledge and their organisation, and the fundamental situations - the *experimental a priori milieu*, that concerns the researcher work preparing the relevant teaching situations, and the *milieu for the contingency* concerning the effective realisation of these situations. In this section we focus on the epistemological model.

- ✓ **About fundamental situations**

For Brousseau, a fundamental situation for a given knowledge ought to permit to generate a family of situations characterised by a set of relationships between student and milieu permitting to establish an adequate relationship to this knowledge.

- ✓ **The need of a model of epistemological *milieu***

To give a definition of what could be an adequate relationship to a given knowledge is not so easy that it could appear at first sight. It is the task of a researcher who attempts to elaborate a model of epistemological milieu:

> "Such a model (written Mi_T) is elaborated taking in account the cultural mathematical knowledge, but is not restricted to it. To elaborate *milieus* consists in grouping problems that do not necessarily obey strictly to the knowledge organisation, thus a conjunction of mathematical, epistemological, and referential practices is necessary. I will add and of identification of knowing. Thus, one has to take into account not only problems for which this knowledge is functional, but also the relationship between these problems, and as far as it is possible, the related knowing (possible actions, intuitions, personal and cultural references) that the student could be able to actualise in the situation. " (Bloch, 2002, p.5)

Our ambition, in this research, was not to elaborate a fundamental situation for a given notion (for us the notion of irrational number), but to attempt to enrich the net of relevant problems for the learning of this notion, leaning on a study (non exhaustive) of « the historical genesis of the knowledge concerning this concept and its ancient or contemporaneous occurrences, its functionalities in mathematics... » (Op. cit. p.7) as well as its links with other fields of human activity (philosophy ; sociology ; history ; psychology ; didactics …), all links that have to be taken in account in the elaboration of an epistemological milieu as defined above. This permits to clearly investigate the way to elaborate the milieu for a teaching situation aiming to integrate this historical genesis and this anthropological perspective. In other words, how to make possible that historical or cultural references, beyond their function of motivation, contribute in a genuine way to the teacher's project of the elaboration by students of knowing coherent and consistent with the involved knowledge. We will give further some elements that we have identified in this research.

III. AN EXAMPLE IN SECONDARY SCHOOL: IRRATIONALITY

III.1. Preliminary: a logical point of view

In a major work of Analytic Philosophy[9], the philosopher and logician Quine support the thesis that attributing a pre logical mentality to natives is wrong; in particular, rather than considering that they have contradictory believes, we have better to bet on an inadequate translation, or in a domestic situation[10], on a linguistic disagreement. In other words, the irrationality or the incoherence of humans is less probable than a non adequate interpretation by the observer of the provided indicators. We have shown (Durand-Guerrier, 1996) an example of the domestic version in mathematics education in order to lift a suspicion of incoherence that might bear on students' responses[11]. Matters concerning contradiction, rationality and irrationality are subjects of study for logicians, either those attempting to elaborate systems accepting contradictory propositions, due to the fact that such propositions are everywhere in ordinary life (e.g. Da Costa, 1977), or those developing theories taking in account simultaneously syntactic, semantic and pragmatic considerations in natural languages[12]. In this perspective, the Model Theory developed by Tarski (1936) offers a relevant theoretical framework to deal with the questions of necessity and contingency, and to treat apparent contradictions (Durand-Guerrier, 2006, 2008).

The project of Granger (1998) is « to consider the sense and the role of irrational in some human works, in some major creations of human spirit, and particularly in sciences. »[13] (Op.cit. p.10). From an author who has devoted his work to description, analysis and promotion of what is rational in human thought, this is not an apology of irrationality, but the testimony of an inscription in « the perspective of an open and dynamic rationality, in order to recognise and delimitate the role of what is positive in irrational. » (Op.cit. p.10). Indeed, Granger considered that « the irrationality, eminently polymorphic, draws in hollows, so saying, the form of rationality (…), and always supposes, at least for analysis, a representation of what it is opposing with." (Op.cit. p.9)

According to us, theses short insights show that the crucial opposition in number theory between rational and irrational number, articulated with the opposition between coherence and contradiction, is a candidate for our exploration.

III.2 Our research hypotheses

Two main hypotheses are structuring our work. The first one is that the problematic of the articulations between various modes of thought, in particular the relationship between Science and Myth, Rationality and Beliefs, is relevant for the study of anthropological fundaments of mathematics. The second one is that, through the intermediary of the genesis of mathematical knowledge, we will be able to achieve an anthropological thought concerning mathematics and their links with the various modes of human thoughts.

III.3 The inscription of Irrationality in our investigation

The term Irrational (in Greek: alogon) has two main significations. First, it means « without a common measure; that cannot be measured as a quotient of two integers ». Second, it means « that is unable to insure the coherence of discourse; illogical ». For Granger (1998) the encounter of irrational numbers in Greece was an example of what he named « the irrational as an obstacle, starting point of the conquest of rationality anew ». This leads to two partly philosophical questions: in what consists really the obstacle? In what consists its resolution? Arsac (1987) supports that the encounter with Irrationality is at the origin of the transformation of mathematics in an hypothetical deductive system. Of course, it is clear that the confrontation to Irrationality only is not sufficient to create anew the conditions of the apparition of the proof, but this invites us to turn toward an interdisciplinary approach of rigor, that we have modestly done in our work. If students of grade 8 or 10 are not a priori able to overcome the epistemological obstacle[14] (indeed, it would be necessary to work along two axes: Euclid Theory of grandeurs; and a real number construction), our weaker hypothesis is that the confrontation of students with a mathematical or an interdisciplinary work about Irrationality could permit them to approach the question of the nature of this obstacle.

IV. OUR DIDACTIC INVESTIGATION

IV.1. General conditions for a didactical situation in our perspective

In coherence with our theoretical exploration, we propose conditions that a didactical situation dedicated to the introduction of an historical and anthropological perspective for a given knowledge in mathematics in secondary school ought to fulfil. 1. The situation leans on a moment well identified of the genealogy of this knowledge. 2. The situation permits to question the formidable efficacy of mathematics to act in real world[15]. 3. The situation fulfils the minimal conditions of a problem situation, in particular favouring framework changes (Douady, 1986). 4. The milieu is rich enough to provide retroactions permitting to go forward in the situation and conditions for an intern validation. 5. From the situation, a contradiction between *a priori* beliefs and constraints from reality would emerge. 6. The situation permits to end up in an institutionalisation of the involved concept in coherence with the curriculum, and of the specific contribution of mathematics to a more general problematic, linked most often to Human and Social Sciences.

IV.2. Brief description of the experiment in grade 8

This experiment took place in December 2000 and January 2001, in an interdisciplinary project. It comprised four sessions in History course (18th century); five sessions in French course, on the thematic of rational and irrational; and four sessions of mathematics that we describe below.

- ✓ First session: construction of a square from a pair of superposable squares with sides of 10 cm, using a minimal number of cuttings with scissors; elaboration of a proof that the figure is actually a square.
- ✓ Second session: synthesis of the proofs elaborated in first session; investigation in order to determine the area of the big square.
- ✓ Third session: enlightening of the fact that the length of the side of the big square is not a decimal number. Emergence of the following question: is it a rational number?
- ✓ Fourth session: elaboration of a proof that $\sqrt{2}$ is not a rational number. Information about the circumstances of this discovery; historical and anthropological aspects; links with what have been done in History and French's courses.

In April 2001, an evaluation has been done through a role game (Pythagoras' Trial) organised by the three teachers involved in the experiment.

IV.3. Some results of the experiment in grade 8

The interdisciplinary work has permitted to make explicit the links, although students have not always perceived them. Concerning mathematics, it is necessary to find a balance between levels of difficulty on the one hand and interest and relevance of the problem on the other hand. This is the case in general for problem situations, but here due to the conceptual ambition it is more acute. Teachers do not wish that their students face difficulties; but the contents, although they do not really exceed the programs, mobilize cognitive capacities hardly required in the ordinary school mathematical work. However, the effective experiment permits to enlighten that most students appreciated this type of problem and were able to provide rich and relevant arguments.

Students have dealt with the following mathematical notions: area of a square by cut-out; property of areas to be additive; units; recognition of equality of two squares constructed by two different methods; calculations on decimal numbers, on rational numbers; interrogation about results given by a calculator. Moreover, they have developed argumentation and deductive reasoning in geometry (for example, justify that a figure is a square), and in the numerical field (it is impossible that the square of a decimal / a rational number be equal to 2). Notice that the last proof is that one using the possible digits of the numerator and the denominator, and *reductio ad absurdum* (or infinite descent).

The analyses of the evaluation (Pythagoras' trial) on the one hand, and of three interviews of students on the other hand, give us *a posteriori* information. The development of the trial seems to indicate that students have understood the arguments; have discussed together, but did not have enough time for a right appropriation of the working of a trial. Here are some arguments developed buy students: "If the diagonal of the square is neither an integer, nor a decimal, nor a rational, he (Pythagoras) has not invented, for this length existed." / "The accusation: it is serious not to reveal this discovery, it is a lost of time -The defence: he will not have been believed. - The accusation: but he had explication! It will have end that he will be believed; he had a theorem." / "If he revealed the irrational numbers, his whole previous theory would have been wrong.- these number frighten - to say these numbers would have caused the end of the world ; it would have disturbed everything." (this student makes a distinction between ordinary people and scientists). / "When he (Pythagoras) said everything is number, he was not lying because at that time, he did not know about the existence of irrational number."

The interviewed students remembered precisely what had been done in the four sessions of mathematics. The link between Irrationality in Mathematics and in French and /or History courses is not done by all of them, but one of them summarized it saying "when we see the superstitions of humans, the sects, it may disrupt the world, and the number too, it may disrupt the world. There is a small link, but it is different."

This project provides an alternative to the aspect "tools" generally devoted to mathematics. Although this aspect "tools" is quite relevant, many teachers perceived it as a reduction of what is mathematics really. This project permits that school mathematics also play their role, beside others disciplines, in the elaboration of elements of human culture, beyond the strictly technical aspects, that an excessive recourse to algorithms tends to reduced it to.

IV.4. Brief description of the experiment in grade 10

The experiment took place in 2002-2003 by an experienced teacher, and in 2006-2007 by a prospective teacher in the frame of the professional dissertation in the Teacher Training Institute (IUFM) in Lyon. It comprised five sessions

- ✓ First session: Introduction of the problematic of incommensurability through the following problem: given a square ABCD, is it possible to find a unit measuring both the side and the diagonal of the square; you may use calculator but not the key of square root. Students worked first in small groups; a square of side 12 cm had been provided; the synthesis was collective in the whole class.

- ✓ Second session: Working on the link between Incommensurability and GCD (Euclid Algorithm) in the whole class.
- ✓ Third session: Proof of the incommensurability of the diagonal and the side of a given square, by *reductio ad absurdum* in the geometric framework.
- ✓ Fourth session: Irrationality of $\sqrt{2}$; approximation by rational numbers.
- ✓ Fifth session: work on texts and documents; realisation of posters.

IV.5. Some results of the experiment in grade 10

In grade 10, the teachers consider that the first four sessions were rich for the following reasons. 1. They give a meaning and a legitimacy to proof, as said a teacher. « Indeed, some students have difficulties to understand the necessity of proof. When we propose a proof for a problem for which they know the result, they do not understand why proving. Here, a debate rose at the first session. Some were convinced of incommensurability of the side and the diagonal of the square, but others were not. The objective of the proof was to convince, to argue. Let us notice the role of *reductio ad absurdum* in the third session; however it is not involved *a priori* in the numerical field to prove irrationality, but in the geometrical situation that permits to prove this incommensurability; moreover this incommensurability has been studied experimentally in the first session (in a geometrical or numerical field, according with the process used by students), that permits to pose the problem in a better way » / 2. "They make links between numerical and geometrical field. Some notions allowing solving the problem have got signification for students as GCD or Euclid algorithm." / 3. "These sessions have permitted an evolution of the vision that students had of mathematics: « we have enlightened the fact that the construction of mathematics did not occur in a linear way but through ruptures ». So they could change their mind that mathematics « go on their own ». As sometimes mathematicians face difficulties to apprehend some notions, students realise that their own difficulties were normal." / 4. "They permit to revised various mathematical notions: GCD – Euclid Algorithm – Pythagoras theorem – rational number …" / 5. "All students have been involved in this work (as well at school as for homework), and interested whatever their level."

CONCLUSION

We thought we have given some evidence (in an *existential* sense) that it is possible in grade 8 and 10 in France to do an interdisciplinary work, structured around a mathematical notion, for which a study, even partial, of the historical genesis permits to enlighten the anthropological dimension in the sense we have defined above. Irrationality appears so as a paradigmatic theme, or even an *ad hoc* theme, of what we aim to develop. That other notions could permit such a work remains for us an open question, but it seems to us that it would be possible to find candidates towards themes common to mathematicians and philosophers, sociologists, historian, without forgetting artists: propositions; infinity; emptiness; space-time; paradox; truth; necessity; transcendence….

In France, there exits *a priori* some institutional niches where organising such a work permitting that mathematics were not only a tool for other disciplines, but took part in the search for links between different ways to understand the world. Nevertheless, it necessitates from teachers a genuine engagement and an enthusiasm that, in general, the schooling institution does not favour.

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[2] LEPS-LIRDHIST : Laboratoire d'Etude du Phénomène Scientifique, EA 4148, équipe Didactique et Histoire des Sciences et des techniques

[3] The "Commission Inter IREM d'Epistémologie et Histoire des Mathématiques" was created on May 10th 1975. Since this date, three meetings and a conference are organised each year. Several national and European summer schools are organised. Its members take part every four years to the international ICME and HPM conferences.

[4] Our translation

[5] See for example Barthes (1957).

[6] Our translation

[7] We refer to the electronic version of the CD-ROM in the Proceedings of the 11th Summer School: Dorier & al. (2002)

[8] Our translation

[9] Quine (1960) Word and object

[10] That means our co speaker

[11] Durand-Guerrier (1996) pp. 276-280

[12] The use of such a perspective in primary and lower secondary education can be found in Durand-Guerrier & al (2006)

[13] Our translation

[14] This notion introduced by Bachelard has been used in mathematic education (Brousseau, 1998)

[15] In reference to a famous quotation attributed to Einstein.

THE IMPLEMENTATION OF THE HISTORY OF MATHEMATICS IN THE NEW CURRICULUM AND TEXTBOOKS IN GREEK SECONDARY EDUCATION

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The official textbooks for the teaching of mathematics in the Greek high school (7^th^-9^th^ grades) include a lot of historical material, following the guidelines of the new curriculum: historical snippets and historically motivated activities, aiming to provide teaching tools for better understanding the mathematics. However, their use is questionable because of serious historical errors, obscurities, or omissions. We support this conclusion by some examples, suggest alternative ways to use this historical material, and outline a more demanding and deep way to use the history of mathematics as a teaching tool that has been implemented in the context of cross-curricular activities.

Key words: historical snippet, mathematics curriculum, cross-curricular, original sources, junior high school.

INTRODUCTION

In the last two decades, there is an internationally increasing interest in introducing a historical dimension in mathematics education (ME), both in didactical research and in the context of educational policy, curriculum design and textbook content. This is reflected in the appearance of several publications, the organization of conferences and meetings, especially in the context of the so-called HPM Group (e.g. Fauvel & van Maanen 2000, Siu & Tzanakis 2004, Katz & Michalowicz 2005, Schubring 2006, Furinghetti *et al* 2006, 2007, Barbin *et al* 2008). In Greece, there has always been an active interest in this area, as early as the late '80s, mainly in didactical research (Fauvel & van Maanen 2000 §11.8, Kastanis & Kritikos 1991, Thomaidis *et al* 2006, Chasapis 2002, 2006) and occasionally in the inclusion of short historical comments in school textbooks. It seems that among other things, the influence of the work of researchers and educators, active in this area, led the Ministry of Education and its associated authorities to become more attentive to what international research and practice suggests concerning the role of the History of Mathematics (HM) in ME. As a result, for the first time in Greece the (new) mathematics curriculum for compulsory education (officially announced in 2002 -see Pedagogical Institute 2002- and implemented via newly written textbooks since 2007) includes so important and extensive references to the didactical integration of the HM into the teaching of mathematics. These references vary from the specific teaching objectives, to the didactical methodology and the textbook content. The following extracts are indicative (Pedagogical Institute 2002 pp.311, 367-369, our translation):

Special objectives: *"..... to reveal the virtue of mathematics (historical evolution of mathematical tools, symbols and notions)."*
Didactical methodology: *"... It is important to provide students with "safety valves" in the pursuit of knowledge; namely, students should be given the possibility to approach a notion in a variety of ways, i.e.:*
-By means of several different representations (using symbols, graphs, tables, geometrical figures);
-In an interdisciplinary way;
-With reference to the HM (the HM is a field rich of ideas to approach a notion didactically)."
Didactical material: *"... Moreover, reference to the great historical moments that step by step have determined the development of mathematics should be included in the mathematics textbooks, so that the student becomes aware of the genesis of the ideas, which is a prerequisite for grasping each subject. It is not necessary that the historical notes appear separately at the end of each §. (If required), they can also be (briefly) presented, at intermediate parts of the text."*

Although these guidelines follow what didactical research seems to suggest nowadays, focusing on the important role HM can play in ME, their actual classroom implementation is far from being satisfactory. More specifically, the authors, who participated in the competition for writing the textbooks according to the new curriculum[1], have tried to follow these guidelines, incorporating in the new mathematics textbooks a great deal of material from the HM in the form of historical notes and associated activities. These notes and activities (called *historical snippets*; Fauvel & van Maanen 2000, ch.7) have different format and colors from the main text and usually contain pictures. Here we examine critically the validity of the material in question and its relevance to what is referred to in the curriculum, by means of specific examples. Then, we proceed to suggest other ways to integrate the HM in teaching, by taking into account some modern trends in this direction.

THE VALIDITY OF THE HISTORICAL TEXTBOOK MATERIAL AND ITS RELEVANCE TO THE CURRICULA SPECIFICATIONS

The quotations from the mathematics curriculum in §1 directly connect the use of the HM with a central issue of teaching and learning: how to pursue and grasp knowledge. Thus historical snippets in the textbooks should not be limited to factual information, but contribute to understanding the notions to be taught (Fauvel & van Maanen 2000, §7.4.1); they should provide the teacher with ideas and material to organize teaching and motivate students to learn. Therefore, they should meet two reasonable requirements: (a) to be mathematically and historically correct; (b) to serve the objectives of the teaching units in which they are incorporated.

Unfortunately, in many cases the historical snippets in the new high school textbooks do not satisfy these requirements; the authors' insistence on restricting the historical material to (often inaccurate and contradictory) biographical information, is a typical case. In general this material is presented in an informal style, inserted in separate boxes in the text, usually with emphasis put on the historical facts, rather than the mathematical exposition. In some cases it also includes related activities (cf. Fauvel & van Maanen 2000, §7.4.1). Table 1 gives a summary of the historical material in the new textbooks:

Table 1

Grade	Number of historical snippets	Percentage of textbook pages covered	Percentage of snippets which include activities	Comments in the teachers book
7	21	11/220 = 5%	5/11 = 45,5%	No comments on the HM
8	9	6/230 = 2.6%	0/6 = 0%	2 additional activities are recommended
9	5	5/240 = 2.1%	2/5 = 40%	10 additional comments covering 12 of the 100 pages (1 activity recommended as an interdisciplinary activity.

We illustrate this material and its weaknesses by means of some indicative examples, mainly from the 7th grade textbook (Vandoulakis et al 2007, Vlamos et al 2007)[2].

Example 1: factual information; no mathematics involved

In the 7th grade textbook, the authors cite 3 different "estimates" of Euclid's lifetime giving contradictory results: p.26: Euclid (330-275 BC); p.147: Euclid (300-275 BC); p.182: Euclid (330-270 BC). Here it is ignored that the only existing valid historical source on this point, is an extract from *Proclus' Commentary* on the First Book of Euclid's *Elements* with no possibility to specify an exact time period (see §4, here). In addition to historical confusion, this note does not serve any of the purposes of introducing HM in teaching as detailed in the new curriculum (cf. § 4 below).

Example 2: factual information; reference to mathematical & scientific results

In the same textbook (p.29), brief information is given on the life and scientific achievements of Eratosthenes in a separate box, in which the life period of Eratosthenes and some of his achievements (e.g. the measurement of the earth's circumference) are simply stated. It is claimed that: Eratosthenes lived from 276BC until 197BC; from 235BC and for 40 years he was director of Alexandria's famous library; at the age of 82, he committed suicide because he became blind. The dates referred to in that note are contradictory, however: Since 276-197=79 and 235-40=195, Eratosthenes lived 3 years less than the age at which he died, and he directed Alexandria's library for two years after his death! This note could include interesting activities in accordance to the regulations of the new curriculum (e.g. the simplicity of the measurement method of the earth's circumference); instead, it is restricted to simply assert the results, which is mystifying, rather than enlightening!

Example 3: fiction, mathematical results and a related mathematical activity

Occasionally, the historical narrative is fictitious. In the 7th grade textbook, historical accuracy is sacrificed in favor of a controversial story, aiming to dramatize an episode from Gauss' childhood (p.75, our translation):

"Sometimes a simple thought of a man is more worthwhile than the whole world's

gold. With some clever ideas battles are gained, monumental pieces of work are done, people become famous and at the same time, science is developed, technology evolves, history is shaped and life changes. Just an example is the "smart addition" that Gauss (Karl Friedrich Gauss 1777-1850) had thought of in a small German village, around 1789, when he started learning about numbers and arithmetical operations in his first year at school. When the teacher asked his students to calculate the sum 1+2+3+...+98+99+100*, little Gauss had found it before the others even started. Then, he wrote on the blackboard:*

(1+100)+(2+99)+(3+98)+...+(48+50)+(50+51)=
101+101+101+...101+101+101=101·50=5.050

Try to calculate in Gauss' way the sum 1+2+3+...+998+999+1000 *and measure the time needed. How much would it have taken if you had calculated it with the normal addition?"*

Firstly, Braunscheweig, where Gauss was born and lived was not a village, but a political and cultural center, capital of a ducat, with approximately 20.000 residents in the late 18th century. Secondly, given that Gauss was characterized as a "child-prodigy" in mathematics from the age of 3, how is it possible that he started learning arithmetical operations as late as 1789, at the age of 12? We do now that Gauss entered the Volksschule (elementary school) in 1784, i.e. at the age of 7, the Gymnasium in 1788 and the Collegium in 1792 (Wussing & Arnold 1978, p.318). Moreover, Gauss died in 1855, not 1850!

In addition, this note starts with an extreme statement, suggesting that mathematical progress is due to a few geniuses, not a collaborative enterprise in which personal skill is harmoniously combined with preceding achievements of the scientific community at the right moment. Thus, it implicitly gives a distorted view of history, which, considered didactically, is expected to discourage rather than engage students in mathematical activities in the classroom. Hence, this example shows lack of relevance of the textbook's historical material with the curriculum objective "*to provide students with 'safety valves' in the pursuit of knowledge*".

Example 4: historical snippets with historically motivated mathematical activity

In the same textbook there is the following activity (p.75, our translation):

ACTIVITY: On a gravestone the following problem is inscribed, whose solution gives the age of the great ancient Greek mathematician Diophantus:

"This tomb holds Diophantus. Ah, how great a marvel! The tomb tells scientifically the measure of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, he clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! Late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life."[3]

This activity, included in the chapter "Equations and Problems", is not accompanied by any query! In the teacher's book, we find (p.53, our translation):

"*A*. 4.2. Problem Solving: *Indicative design of the material of this unit*. *1 teaching hour*. The suggested activity aims to understand: The notions used in problems, their solutions, as well as, the solution process we follow [*Answer*: Diophantus lived for 74 years]".

If this requests the formulation of an appropriate equation for Diophantus' age *x*, then the data in the epigram imply:

$$\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4=x \Leftrightarrow x=84$$

Historically it may be asked where lies the gravestone, in which this problem can be found? We do know that this story is included in the *Palatine Anthology,* of the Byzantine era and there is no other reliable evidence for it. Didactically, the question is whether 7th graders are able to formulate and solve this equation, given that solving such equations is taught in the 8th grade! This clearly shows that some historical notes are related neither to the mathematics of the textbook unit in which they are included, nor to the cognitive level of the students to whom they are addressed.

The same epigram appears in an introductory note in the chapter on "*Equations and inequalities*" of the 8th grade textbook with the following comments (Vlamos et al 2007, p.120, our translation):

"*...From his* [Diophantus'] *13 pieces of work only 10 had been found (6 in Greek manuscripts and 4 in Arabic translation). The most famous of his works is the "**Arithmetika**" (6 books). It is the most ancient Greek work in which for the first time a variable is used in problem solving…When he died, ...his students composed a riddle and wrote it on his grave, upon his request. Here is Diophantus Epigram…*"

According to Diophantus' own statement, *Arithmetika* were divided into 13 "books"; 6 have been preserved in the Greek original and 4 in Arabic translation of the 9th century discovered in the late 1960's. We also know another of Diophantus' works - "*On polygonal numbers"* – only fragments of which survive. Therefore, in the textbook, confusion is made between the 13 books of "*Arithmetika*" and the total number of his works.

SOME CONCLUSIONS

All examples in §2 concern historical errors (there are still more, simply reinforcing the bad impression one gets from the textbooks' historical snippets) that nevertheless, could easily be corrected in a new textbook edition, though it is strange that they have not been avoided. It seems as if they were written hurriedly and without further check, mainly aiming to satisfy the relevant term of the announcement of the textbook writing competition and not to introduce a historical dimension in teaching.

The main characteristic of the historical material incorporated in the new mathematics textbooks is the large amount of information and the rich illustrations, without however some methodological hints of how to benefit didactically from this material. Although, the corresponding suggestions and instructions in the teacher's book in general emphasize the positive contribution of the HM, the way this could be realized is left to the initiative and ideas of the teacher, with reference to the relevant bibliography. This is what can be concluded by simply reading the corresponding instructions given in the teacher's books. E.g., the teacher's book for the 7th grade mentions that:

"*In some sections, there are historical notes, which intend to stimulate the student interest and love for Mathematics and to inform them on the historical development of mathematical thinking. Their use in teaching depends on the initiative and the ideas developed by the teachers*" (Vandoulakis et al 2007, p.31, our translation)

In the teacher's book for the 9th grade this issue is detailed more:

*"*In some units there are topics from the HM intended to give the description of the problem that has been posed and the presentation of the conceptual tools applied to solve them. These topics, with the accompanying questions, aim to exploit the HM in the best possible way. Integrating the HM in teaching has become the subject of systematic studies at an international level. The positive contribution of the HM is corroborated in three groups of arguments:

(a) It stimulates students' interest and contributes to the development of a positive attitude towards mathematics.

(b) It reveals and stresses the human nature of the mathematical activity throughout history.

(c) It contributes to the understanding of mathematical concepts and problems, revealing not only the context and circumstances in which they originated, but also the conditions of their development.

These topics *[from the HM and the accompanying questions]*, together with those points raised in the teacher's book, should not be considered as complete studies; it is for this reason that references to the literature are given for those teachers and students who will have a special interest.*" (Argyrakis et al 2007, pp.10-11, our translation)*

Remark: Points (a)-(c) form part of the arguments for integrating HM in ME, put forward more systematically in Fauvel & van Maanen 2000, §7.2 (particularly §§(a1), (c1), (d1).

To introduce a historical dimension in the teaching of mathematics, based on the interest, initiative and ideas of teachers, needs extra teaching time of course. But, apart from the usual obligation to cover the school material (a very difficult problem in itself!), teachers have also to cope with the innovations of the new curriculum, like group-cooperative teaching based on learning activities, or an interdisciplinary approach to mathematics. Adding the introduction of a historical dimension to the benefit of both teachers and students, requires additional support in the form of detailed guidelines (e.g. examples serving to illustrate how history could be integrated into teaching), extensive references for further reading and availability of relevant source material. Unfortunately, existing source material is limited (especially in Greece), a key issue already stressed several years ago (Fauvel & van Maanen 2000, p.212). In addition, from the evidence cited here, it is clear that the material of the new textbooks is not the most appropriate and valid guide in this direction. Therefore, it seems that high school mathematics teachers are not given any real motivation to take up the initiative to benefit from the new textbooks' historical material. In the next section, we examine whether the available historical snippets (of course free from mistakes and contradictory information) can contribute positively to the teaching of high school mathematics.

USING HISTORICAL SNIPPETS IN CROSS-CURRICULAR ACTIVITIES

The errors in the historical notes of §2 indicate that integrating the HM in ME is a demanding activity, presuming, not only mathematical knowledge and the ability to approach, read and interpret the historical sources, but also to cross-check facts, to conclude and narrate. This seems to suggest cross-curricular activities as a privileged framework in this connection. Fortunately, such activities form an integral part of the new curricula and high school textbooks in Greece and a good example in this context could be the determination of Euclid's lifetime. As mentioned in §2, the only valid historical source on this point comes from Proclus, who lived in the 5th century A.D. In his *Commentary* on the 1st Book of Euclid's Elements, he writes:

"[Euclid] *lived in the time of Ptolemy the First, for Archimedes, who lived after the time of the first Ptolemy mentions Euclid. It is also reported that Ptolemy once asked Euclid if there was not a shorter road to geometry than through the Elements, and Euclid answered that there was no royal road to geometry. He was therefore later than Plato's group, but earlier than Eratosthenes and Archimedes, for these two men were contemporaries, as Eratosthenes somewhere says.*" (Morrow 1970, pp.56-57)

This is a nice extract for an activity, combining mathematics, history and language (for Greek students). Translating the ancient text into modern Greek, collecting information regarding the persons involved, studying more the historical period in which they lived, could be a student activity to provide material for further discussion in the classroom, which could lead to the following conclusion:

We know that Ptolemy the First, a general of Alexander the Great had been the satrap of Egypt from 323 to 305 B.C., and its king from 304 to 283, and that Archimedes lived from 287 to 212 BC. Proclus cites the dialogue of Euclid with Ptolemy the First and informs us that he was older than Archimedes. Therefore, we can specify that the period in which Euclid was active is very close to 300 BC.

This activity has interesting didactical extensions and could lead to illuminating discussions on the concept of mathematical proof: The method and logical arguments that led, from historical sources to the above historical conclusion, can be paralleled to those used to justify a general mathematical result from definitions, axioms and previously proven theorems. Hints can also be given for the specific characteristics of theoretical geometry that led Ptolemy to ask Euclid for a "short" learning path to it. Similarly, ancient texts on Eratosthenes' life and work could be used, with emphasis on the measure of the earth's circumference (Thomaidis & Poulos 2006, p.110).

The cross-curricular activities could be also disconnected from the teaching in a conventional classroom and be realized more efficiently in the context of parallel school events, like the formation of a group of students, who, under the teachers' supervision and help, read mathematical works. For instance, the study of the book by Tent (2006) could have more essential pedagogical and didactical results than the historical note about Gauss, mentioned in § 2.

ANCIENT GREEK MATHEMATICAL TEXTS IN THE TEACHING OF EUCLIDEAN GEOMETRY IN HIGH SCHOOL: A CROSS-CURRICULAR APPROACH

In this section, we present some elements of an approach to integrate the HM in teaching mathematics, which is more demanding and deep, than the use of historical snippets; namely the use of original texts in carefully designed worksheets, implemented in cross-curricular activities (Fauvel & van Maanen 2000, ch.9).

We developed a cross-curricular activity in a class of 50 10th-graders (15-16 year old students; 25 girls and 25 boys), for 10 2-hour sessions in which the teachers of mathematics, ancient Greek language and history were involved with alternating interventions. To this end excerpts from Euclid's *Elements* and Proclus' *Commentary*, have been used to construct 4 worksheets that were subsequently used in the classroom. They concern: (a) different proofs of the equality of the angles in an isosceles triangle as they appear in Euclid, Proclus and Pappus; (b) the construction of the bisector of an angle; (c) the triangle inequality for the sides of a triangle; (d) the sum of the angles of a triangle.

This activity aimed to (i) integrate original texts in teaching Euclidean Geometry for 16-years old students in the context of a cross-curricular approach; (ii) to create a new didactical environment and accordingly explore the realization of specific teaching aims, namely, "initiation in mathematical proof", and "development of critical thinking". More specifically, by the chosen excerpts and the questions addressed to the students, we sought to examine whether the students (i) share the criticism of the ancient philosophers against Euclid, (ii) understand the expediency of giving different proofs for the same geometrical proposition, particularly for obvious properties of geometric figures (as Proclus did while defending Euclid) and (iii) understand the expediency of mathematical proof in general. Under the teachers' supervision, students analyzed ancient texts mathematically, linguistically and historically, with focus on formulating mathematical, linguistic and historical questions emerging from the analysis of texts, and classroom discussion of students' point of view on them.

The worksheets were structured as follows: (a) Ancient Greek mathematical text; (b) Request to read and translate the text; (c) Questions on the text: 2 to 3; (d) Homework: 1 or 2 assignments.

Remarks: (1) Three of the worksheets contained 2 excerpts, with this structure for each excerpt; the fourth included 4 excerpts. Due to lack of space, we outline this approach for worksheet No1. (2) The discussions in the classroom were videotaped. Students' answers below refer to questions raised in the classroom (Q1-Q3 below) and come from the analysis of videotapes and the teachers' hand-notes.

Worksheet No1

Excerpts: (i) Euclid "*Elements*" *Book I*, *prop.V*: equality of the basis angles of an isosceles triangle[4]. (ii) Proclus' "*Commentary*", pp. 248, 250: Alternative proofs of the proposition (by Proclus and Pappus)[5].

Questions: (1) Find the corresponding theorem in the geometry textbook.

(2) Find similarities & differences between Euclid's and the textbook's proofs.

Homework: (1) Translate the ancient text keeping to Euclid's spirit as close as possible (e.g. avoid terminology and notation not used by Euclid).

(2) Get information on Euclid and his *Elements* using encyclopedias or other resources.

(3) Translate Proclus' text to modern Greek.

(4) Find similarities and differences among Euclid's, Proclus' and Pappus' proofs.

(5) Try to explain why all ancient proofs are different from that in the textbook[6].

Classroom discussion on the following questions:

Q1. In your opinion, why did Euclid give a complicated proof?

Q2. Why did the ancients avoid using the bisector of the angle at the top vertex? How it can be ensured that the usual construction (by ruler and compass) of the bisector of an angle, does indeed bisect the angle?

Q3. Comment on Proclus' and Pappus' proofs.

Some of students' responses
*On **Q1, Q2***:

(i) Euclid wanted to impress his readers, because when scientists do complicated things, their authority increases.

(ii) Euclid wanted to show how to use the triangles' equality criteria.

(iii) Euclid wants a theoretical, not a practical proof. Bisecting an angle is a practical issue and is not accurate. This construction is naïve, possible for all people, because it is like folding in two a piece of paper.

(iv) Euclid could not draw the bisector accurately; he could not prove that the two angles are equal. The bisector concept had not been discovered yet.

(v) Euclid wanted to exploit that particular proof in order to prove other properties that exist in that particular figure.

*On **Q3** (for Pappus' proof)*:

(i) It looks like proofs that we gave at the elementary school.

(ii) It is a proof appropriate for babies(!)[7]

(iii) It is more difficult; it requires more thinking (it is more probable that we make a mistake).

(iv) It is adapted to practice, whereas, Proclus' and Euclid's proofs have elements of logic and scientific reasoning.

Remarks on methodological issues concerning cross-curricular activities:

(1) A cross-curricular approach to original texts helped to face important issues concerning translation & interpretation and placed original texts in the appropriate historical context.

(2) The original texts and the translation process led to etymological comments on the origin, meaning and accurateness of mathematical terminology.

(3) The clarity and conciseness of ancient Greek mathematical language was revealed by connecting two apparently disjoint disciplines, namely, the study of ancient Greek language and mathematics.

Some results: The above brief comments, and the analysis of the discussion in the classroom stimulated by the study of the other three worksheets, seems to suggest some interesting conclusions:

(a) Studying original texts created a new didactical environment, in which students actively participated in the classroom discourse and exhibited a positive attitude towards the subject under consideration, which never happens in conventional teaching of geometry (this was particularly clear in the critical discussions on worksheet No3 on the triangle inequality and the Stoics' objections reported by Proclus, that tried to ridicule Euclid).

(b) Students' commented that this activity led them to a more global understanding of what Euclidean geometry really is (e.g. see answers (ii) and (v) to Q2).

(c) The variety of students' answers and contradictions among them, that were produced by studying original texts reveal the number of factors that influence the understanding of metamathematical concepts, like the concept of proof (e.g. compare answers to Q3; (i) & (ii) to (iii)).

(d) Critical thinking not only requires the technical ability to formulate particular proofs, but also more general abilities to globally conceive notions, to formulate correct assertions etc (e.g. see answers (iii) to Q3 and (iv) to Q2).

(e) The requirements brought up by studying original texts, link the didactical aims of learning particular mathematical concepts and theories, with wider pedagogical aims of teaching mathematics (raising metamathematical issues, access to philosophical & epistemological concepts, links to the historical & cultural tradition etc- e.g. see answers (i), (iii) and (iv) to Q2).

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[1] In Greece, there is only one textbook in each grade of primary or secondary education, imposed by state regulation as a result of a public competition for writing these textbooks. This concerns all subjects, not only mathematics.

[2] In Greece, grades 1 to 9 constitute compulsory education: the elementary school (grades 1-6; students 6-12 year-old) and the "gymnasium" (junior high-school, grades 7-9, students 13-15 year-old). There are essentially no historical aspects in the elementary school textbooks; hence we restrict the discussion to junior high school.

[3] See Cuomo 200, p.245.

[4] English translation in Heath 1956, pp.251-252

[5] English translation in Morrow 1970, pp.193-195.

[6] In the textbook, the angle at the top vertex is bisected and the two resulting triangles are shown to be equal.

[7] In Pappus' proof an isosceles triangle is turned and the resulting triangle is shown to be equal to the initial one.

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