ABDUCTION AND THE EXPLANATION OF ANOMALIES: THE CASE OF PROOF BY CONTRADICTION[♠]

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Some difficulties with proof by contradiction seem to be overcome when students spontaneously produce indirect argumentation. In this paper, we explore this issue and discuss some differences between indirect argumentation and proof by contradiction. We will highlight how an abductive process, involved in generating some indirect argumentation, can have an important role in explaining the absurd proposition, in filling the gap between the final contradiction and the statement to be proved and in the treatment of impossible mathematical objects.

Key words: proof, argumentation, abduction, proof by contradiction, indirect argumentation.

INTRODUCTION

The relationship between argumentation and proof constitutes a main issue in mathematics education. Research studies have been based on different theoretical assumptions, proposing different approaches and consequently different didactical implications (Mariotti, 2006). In some studies (see, for example, Duval, 1992-93), a distance between argumentation and proof is claimed, while in others, without forgetting the differences, the focus is put on the analogies between the two processes and their possible didactical implications (Garuti, Boero & Lemut, 1998; Garuti & al., 1996). As a consequence, the authors hold the importance for students to deal with generating conjectures, and highlight that this activity can promote some processes that are relevant in developing students' competences in mathematical proof.

Elaborating on this first hypothesis, concerning the continuity between the argumentation supporting the formulation of a conjecture and the proof subsequently produced, Pedemonte (2002) developed the theoretical construct of *Cognity Unity* in order to describe the relationship (continuity or break) between the argumentation process and the related mathematical proof in the activity of conjecture's production.

In this paper, we aim to investigate the relationships between argumentation and proof in the case of proof by contradiction. The reference to the framework of *Cognitive Unity* is of the interest for this study for the following reason. Although important difficulties have been identified in relation to this type of proof (see Antonini & Mariotti, 2008; 2007; Mariotti & Antonini, 2006; Antonini, 2004;

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Stylianides, Stylianides & Philippou, 2004; Wy Yu, Lin & Lee, 2003; Thompson, 1996; Leron 1985), in the literature we find evidence of arguments, spontaneously produced by students, that can be considered very close to proof by contradiction (see Antonini 2003; Reid & Dobbin, 1998; Thompson, 1996; Freudenthal, 1973; Polya, 1945). In fact, as reported by Freudenthal:

> "*The indirect proof is a very common activity ('Peter is at home since otherwise the door would not be locked'). A child who is left to himself with a problem, starts to reason spontaneously '... if it were not so, it would happen that...'* " (Freudenthal, 1973, p. 629)

We call *indirect arguments* the arguments of the form '*if it were not so, it would happen that…*'. Indirect arguments seem to be more like to appear in the solution of open-ended problems, as a natural way of thinking in generating conjectures, when one needs to convince oneself that a statement is true, or to understand because a statement is true.

Therefore, it is seems important to study differences and analogies between proof by contradiction and indirect argumentation, and this is what we are going to do in the following sections.

DIFFICUTIES WITH PROOF BY CONTRADICTION

According with the terminology of the model presented in (Antonini & Mariotti, 2008, 2007), given a statement S, that we called a *principal statement*, a proof by contradiction consists in a couple of proofs: a direct proof of another statement S*, that we call the *secondary statement*, in which the hypotheses contain the negation of S and the thesis is a contradiction (or a part of it); and a *meta-theorem* stating the logical equivalence between the two statements, the *principal* and the *secondary*. Here, we analyse two aspects and their relationships: the link between the *principal statement* and the contradiction achieved through the proof of the *secondary statement*; the treatment of impossible mathematical objects in both the argumentation and the proof.

The link between the contradiction and the *principal statement*

The link between the final contradiction and the principal statement is a source of difficulties for students (see Antonini & Mariotti, 2008). It can happen that such difficulties are openly shown when they appear astonished and disoriented after the deduction of an absurd proposition. This is the case for example of Fabio, a university student (last year of the degree in Physics), who explains very well this type of difficulty:

> Fabio: *Yes, there are two gaps, an initial gap and a final gap. Neither does the initial gap is comfortable: why do I have to start from something that is not? […] However, the final gap is the worst, […] it is a logical gap, an act of faith that I must do, a sacrifice I make. The gaps, the sacrifices, if they are small I can do them, when they all add up they are too big. My whole argument converges towards the sacrifice of the logical jump of exclusion, absurdity or exclusion… what is not, not the direct thing. Everything is fine,*

but when I have to link back… [Italian: "Tutto il mio discorso converge verso il sacrificio del salto logico dell'esclusione, assurdo o esclusione… ciò che non è, non la cosa diretta. Va tutto bene, ma quando mi devo ricollegare..."]

Fabio identifies two gaps (he speaks also of a "*jump*"!) in a proof by contradiction: an initial gap and a final gap. According to our model, the initial gap corresponds to the transition from the statement S to the proof of S*, and the final gap corresponds to the opposite move, from the proof of S* to the conclusion that S is proved. The perception of these gaps makes Fabio feel unsatisfied, as if something were missing. In fact, he can accept the proof but he is not convinced, as he says it is "*an act of faith that must be done*".

The treatment of impossible mathematical objects

It may happen that, at the beginning of a proof by contradiction, some of the mathematical objects have some characteristics that are absurd and strange, in an evident way. These mathematical objects are proved to be impossible in some theory. For this reasons, difficulties can emerge in the treatment of these absurd objects. As discussed in (Antonini & Mariotti, 2008; Mariotti & Antonini, 2006) difficulties may occur in the construction of the proof of S*, but difficulties may also emerge after the proof of S* is achieved, when absurd objects have to be discarded. In fact, at the end of a proof of S*, once a contradiction is deduced, one has to realize that some of the objects involved do not exist; actually, they have never existed. As explained by Leron:

"*In indirect proofs […] something strange happens to the 'reality' of these objects. We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards 'destroying' this world, proving it is indeed false and impossible. We are thus involved in an act of* mathematical destruction*, not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered. What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created? I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick - a sorcery - that has been played on us.*" (Leron, 1985, p. 323).

Our research interest is in exploring whether and how these difficulties may be overcome when students spontaneously produce indirect argumentation. Two elements seem important to take into account: on the one hand the indirect argumentation as a product and its differences with a proof by contradiction, on the other hand the processes involved in producing the argumentation (see also Antonini, 2008). In this paper we focus on the hypotheses that in many cases the students try to fill the gap between the contradiction and the statement in order to re-establish a link and at the same time to give a new meaning to the "objects of the *impossible world*", so that they can be modified without being discarded.

THE ABDUCTIVE PROCESS

Abduction is one of the main creative processes in scientific activities (Peirce, 1960). Magnani defines abduction as

> "*the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated*" (Magnani, 2001, pp. 17-18).

The main characteristic of abduction is that of deriving a new statement that has the power of enlightening the relationship between the observation and what is known.

Many studies in mathematics education have dealt with abductive processes in students thinking: in problem-solving activities (Cifarelli, 1999), in generation of conjectures (Ferrando, 2005; Arzarello et al., 2002; Arzarello et al., 1998), argumentation and proofs (Pedemonte, 2007).

In this paper, through the analysis of a case study, we will show how an abductive process could assume a fundamental role in the production of indirect argumentation. Through an abduction a new statement is produced that has no logical need but allows one to make sense of the absurd and strange proposition and, in this way, to overcome the gap between the contradiction and the *principal statement.*

A CASE STUDY

The following open-ended problem was proposed to Paolo and Riccardo (grade 13), two students that, according to the evaluation of their teachers, are high achievers.

Problem: What can you say about the angle formed by two angle-bisectors in a triangle?

What follows is an excerpt of their interview. After a phase of exploration, the students generated the conjecture that the angle S (figure 1) is obtuse. Afterwards, the students started to explore the possibility that the angle S might be a right angle.

61 P: As far as 90, it would be necessary that both K and H are 90 degrees, then K/2 = 45, H/2 = 45...180 minus 90 and 90 degrees.

62 I: In fact, it is sufficient that the sum is 90 degrees, that K/2 + H/2 is 90.

63 R: Yes, but it cannot be.

64 P: Yes, but it would mean that K+H is ... a square […]

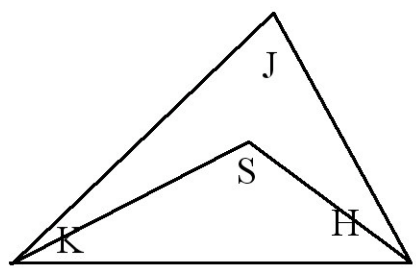

Figure 1: The angle between two angle bisectors in a triangle.

65 R: It surely should be a square, or a parallelogram

66 P: (K-H)/2 would mean that […] K+H is 180 degrees...

67 R: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle.

[…]

80 R: [the angle] is not 90 degrees because I would have a quadrilateral, in fact the sum of the two angles would be already 180, without the third angle. Then the only possible case is that I have a quadrilateral, that is, the sum of the angles is 360.

The episode can be subdivided in three parts: the development of a first argumentation (61-63), the introduction of a new figure, the parallelogram (64-67), the production of the final argumentation (80). This last argumentation is expressed by Riccardo, after the students are explicitly asked to write a mathematical proof.

The argumentation developed in the first part (61-63) is indirect: assuming that the angle between two angle bisectors of a triangle is a right angle, the students deduce a proposition that contradicts a well known theorem of Euclidean Geometry. From the logical point of view, the deduction of the contradiction would be sufficient to prove that this triangle does not exist, or, equivalently, that the angle S is not right, thus concluding the argumentation. But, although convinced that the angle S cannot be a right angle, the students do not feel that the argumentation is concluded and they look for an explanation for the anomalous situation. In fact, the subsequent part (64-67) seems to have the goal to complete the argumentation; in particular, the students seem to look for an explanation to the false proposition "K+H=180° ". An explanation is found by formulating a new hypothesis: the figure is not a triangle, it is a parallelogram. In this case, it is true that the sum of two adjacent angles (K+H) is 180. In search of an explanation the original triangle fades becoming for the students an indeterminate figure that have to be determined in order to eliminate the anomalous consequences. In 67, Riccardo makes clear that the figure can be transformed during the argumentation. His expression "*surely it is not a triangle*" means "this figure is not a triangle" and it must be something else. Differently, in a proof by contradiction, as the proof that could arise from the first part of the argumentation (61-64), the figure is well determined, it is a triangle and it is not possible to modify it. Once a contradiction is deduced, it is proved that this figure does not exist. In this case, the triangle would be part of the "false, impossible world" and it would have had a temporarily role: at the end of the proof we know that it does not exist. Actually, it has never existed.

When the new case is selected and because this new case can solve the anomaly, Paolo and Riccardo seem to be satisfied. In 80, Riccardo summarizes the argumentation in what for him is a mathematical proof. The fact that the angle S is not right is not proved by contradiction but is based on the analysis of different cases: triangle, square, parallelogram. The figure is determined, and it is not a triangle, as we have thought at the beginning of the argumentation. This argument seems very convincing for students, more than the argument based on deriving a contradiction.

The key point in the development of the argumentation is the generation of the new case that is the identification of the parallelogram. This process can be classified as an abduction, in fact an explanatory hypothesis is produced and evaluated, as Riccardo says "[the quadrilateral] it is the only possible case".

The assumption of the parallelogram transforms a false into a true proposition. This argument allows students to overcome some of the difficulties that might be raised by a proof by contradiction (figure 2). In particular:

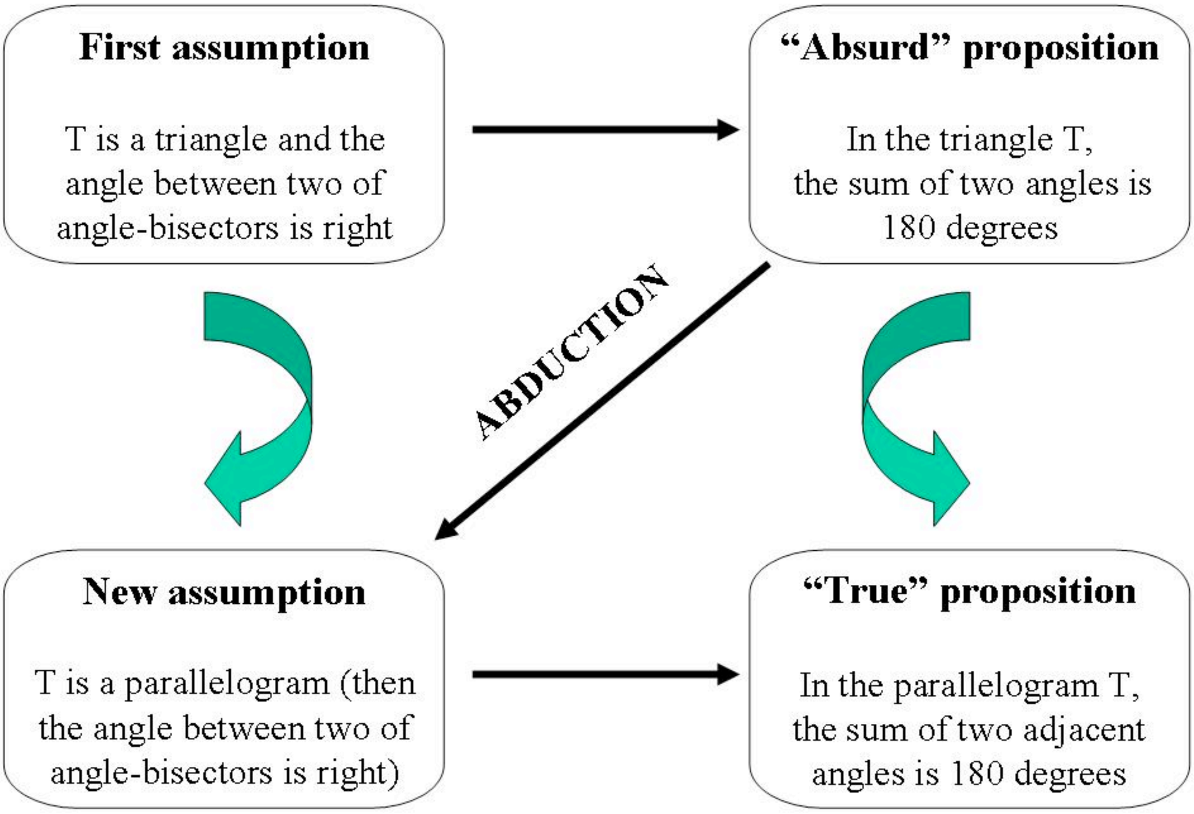

Figure 2: An abductive process in an indirect argumentation

- The false proposition - "in a triangle the sum of two angles is 180°" – becomes a true proposition related to the new explanatory hypothesis (in a parallelogram the sum of adjacent angles is 180°);
- The mathematical object (the triangle) is considered an indeterminate object that is identified only through the abduction with the goal to explain the anomaly. Then the mathematical object is changed and not discarded as it happens in a proof by contradiction. The problem of treatment of mathematical object at the end of proof by contradiction highlighted by Leron (1985) is bypassed.

- Differently to what happens with proof by contradiction, a link, that is not only logical, between the *secondary statement* and the *principal statement*, is constructed: it is not possible that S is right because otherwise the triangle would become a quadrilateral.

As the previous example shows, in geometry, the identification of the case that can explain the anomaly and allow getting out of the "impossible world" seems to be related to the transformation of figures. Most of the students asked to solve the problem of angle bisectors provided arguments based on transforming the triangle in a quadrilateral or in two parallel lines.

Further researches are necessary to corroborate this hypothesis and investigate whether it can be extended to other context. In fact we hypothesize that also in contexts other than Geometry abduction can be for students the key to come out from the anomalous situation that occurs in proof by contradiction. In order to support this extension to other contexts, we report now a short episode concerning the algebra domain.

ABDUCTION AND PROOF BY CONTRADICTION IN ALGEBRA: AN EXAMPLE

In a questionnaire proposed to 68 secondary school students (grade 10, 11, 12) and 19 university students, a proof by contradiction of the incommensurability of the diagonal of a square with its side was presented. We aimed to investigate the recognition and the acceptability of this type of proof. In the presented proof, it is assumed that the ratio is a rational number, expressed by the fraction m/n where m and n are two natural numbers (with n different from 0). Then it is deduced that the number n is both odd and even. The students were asked to choose one of the following answers to explain what it is possible to conclude:

a) This is not a proof

b) There is a mistake in some passages, but I can not identify it

c) There is a mistake, that is (specify the error): ....

d) We have not proved anything, because being even or odd has nothing to do with what we wanted to prove

e) We have proved what we wanted, in fact:………………………………………

f) Other (specify):

The 25 per cent of the students gave the correct answers and the 35 per cent chose the answer d). This expresses the feeling that something is missing and let us suppose the need to see a link between the contradiction and the statement. A hint in this direction comes from one of the answers. One student (grade 12) marked the correct answer and explained:

> "*we have proved what we wanted in fact one of the two numbers [the number n] is not a natural number and then the ratio is not a ratio between two natural numbers*"

The argument provided does not refer to what could be recognized as the *meta-theorem*, explaining the logical equivalence between the *principal statement* and the *secondary statement*, and thus rejecting the existence of the mathematical object m/n.

Differently, this student does not reject the initial assumption that the ratio is rational from the contradiction "n is even and odd", rather he changes the nature of the number n coherently (in his opinion) with the deduced proposition. If m/n is not a rational number, as we have believed before, everything is explained.

Inferring the explaining hypothesis that number n, odd and even at the same time, is not a natural number is the product of an abduction. The hypothesis that n is not a natural number can explain the anomaly "n is odd and even" and, at the same time, it offers a link between the deduced proposition and the principal statement: n is not a natural number and then the ratio m/n is not a rational number. A link between the contradiction and the statement is now established and the proof can be accepted.

CONCLUSIONS

Main difficulties with proof by contradiction are related to the link between the contradiction and the statement to be proved, to the treatment of the impossible mathematical objects during the construction of the proof and at the end, to the need of discarding the mathematical objects involved in the proof of the *secondary statement*. The feeling of frustration that may emerge at the end of a proof by contradiction, as clearly expressed by Fabio's words, is accompanied by the need of giving a meaning to the absurd proposition, the need of establishing a stronger link with the principal statement and adjusting the "false, impossible world".

The analysis of the episodes proposed above shows how abductive processes may be mobilized to produce explanatory hypotheses. The system of relationships represented in the diagram of figure 2 shows the key role of the abductice process and highlights some differences between indirect argumentation and proof by contradiction.

Interpreting these results in terms of Cognitive Unity leads us to point out the distance between indirect argumentation as it is spontaneously developed and the scheme of a proof by contradiction. In particular, it clearly appears the distance between the *meta-theorem* - providing the equivalence between the *principal* and the *secondary statement* - and the abductive process that might emerge in an indirect argumentation. The question rises whether and how such distance can be filled through an appropriate didactical intervention.

Of course, further investigation is necessary to better understand the differences between argumentation and proof by contradiction and to identify and analyse other processes that could be important for the production and the development of indirect argumentation.

We think that the comprehension of these processes is fundamental for teachers to identify, explain and treat students' difficulties with proof. We also believe that indirect argumentation, even if it presents significant differences with proof by contradiction, should be promoted, in particular through open-ended tasks. As Thompson writes:

"*If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure.*" (Thompson 1996, p.480)

As regards the transition from the argumentation to proof by contradiction, further researches are necessary to identify the tools to construct the didactical activity to face the gaps and promote the acceptability of method of proof by contradiction.

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CONJECTURING AND PROVING IN DYNAMIC GEOMETRY: THE ELABORATION OF SOME RESEARCH HYPOTHESES

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Research has shown that the tools provided by dynamic geometry systems impact students' approach to investigating open problems in Euclidean geometry. We particularly focus on types of processes that might be induced by certain uses of tools available in Cabri. Building on the work of Arzarello (Arzarello et al., 1998) and Olivero (1999, 2002), we have conceived a model describing some cognitive processes that may occur during the production of conjectures and proofs in a dynamic geometry environment and that might be related to the use of specific dragging schemes. Moreover, we hypothesize that such cognitive processes could be induced by introducing students to the use of dragging schemes.

Key words: conjecturing, dynamic geometry, dragging schemes, abductive processes, cognitive unity

INTRODUCTION

The contribution of a DGE to students' reasoning and proving is particularly evident during the investigation of open problems, since this process involves making conjectures (Mariotti, 2006). Instead of a static-conjecture built in a paper-and-pencil environment in a DGE a dynamic-conjecture [1] can be developed. Moreover, in a DGE, the invariant geometrical properties of a construction, which lead to conjectures, can easily be grasped. An interesting question is: what kind of support can a DGE provide first during the development of a conjecture and then during the production of a proof? The answer seems to depend on the nature of the problem. On one hand the ease to immediately grasp certain invariants seems to inhibit some argumentation processes that lead to finding useful elements for the construction of a proof. On the other hand, research has shown that a DGE can foster the learners' constructions and ways of thinking, and that it can help overcome some cognitive difficulties that students encounter with conjecturing and proving (e.g. Noss & Hoyles, 1996; Mariotti, 2002; De Villiers, 2004).

Building on the work of Olivero and Arzarello (Olivero, 1999; Arzarello et al., 1998), we have conceived a model of cognitive processes that can occur during the conjecturing stage of open problem investigations in a DGE. Through a qualitative study, our final goal is to give a detailed description of some cognitive processes related to conjecturing and proving, and of how a DGE might foster such processes, thus providing a base for further research and for the development of new curricular activities.

ORIGIN OF OUR HYPOTHESES

In the following paragraphs we will briefly outline the theoretical framework which the ideas are embedded in.

Semiotic Mediation and Semiotic Potential of an Artifact

A DGE like Cabri, which contains "objects" such as points, lines, circles, and ways to "manipulate" the objects, is a microworld (Papert, 1980; Balacheff & Kaput, 1996) built to resemble the mathematical world of Euclidean geometry. A key aspect of microworlds in mathematics education is that the "objects" included offer the opportunity for the user to experiment directly with the "mathematical objects" (Mariotti, 2005, 2006), because the logical reasoning behind the objects in the microworld is designed to be the same as that behind the real mathematical objects that they represent.

Recent research has developed the ideas of *tool of semiotic mediation* and of *semiotic potential of an artifact*:

> "...any artifact will be referred to as a tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention" (Bartolini Bussi & Mariotti, 2008).

Computers in general, and a DGE in particular, are considered to be tools of semiotic mediation (Mariotti, 2006; Bartolini Bussi & Mariotti, 2008). However, the mediation can occur successfully only if their semiotic potential is exploited. Therefore it becomes necessary to study ways that foster exploitation of such potential. This was a main goal we had in mind when we started developing our hypotheses.

A First Theoretical Model and the Dragging Schemes

The dragging tool can be activated by the user, through the mouse. It can determine the motion of different objects in fundamentally two ways: direct motion, and indirect motion. The *direct motion* of a base-element (for instance a point), that is an element from which the construction originates, represents the variation of this element in the plane. The *indirect motion* of an element occurs when a construction has been accomplished. In this case dragging the base-points will determine the motion of the new elements obtained through the construction. The use of dragging allows one to feel "motion dependency", which can be interpreted in terms of logical dependency within the geometrical context (Mariotti, 2002, p. 716). Starting from these phenomenological perspectives, a refined analysis of the dragging tool can highlight its semiotic potential that can be exploited by the teacher in school practice.

The use of Cabri in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between hypothesis and thesis. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the Cabri figure directs the way of transforming and observing the screen image (Talmon & Yerushalmy, 2004). At the same time, that consciousness is needed to exploit some of the facilities offered by the software, like the ‘locus of points’ or ‘point on object’. Such a consciousness is strictly related to the possibility of exploiting the heuristic potential of a DGE (Mariotti, 2006).

The theoretical model presented by Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 1998, 2002) addresses expert solvers’ production of conjectures, and how abduction marks the transition from the conjecturing to the proving phase, when a passage from “ascending control” to “descending control” occurs. Abduction guides the transition, in that it seems to be key in allowing solvers to write conjectures in a logical 'if…then' form, a statement which is now ready to be proved. Arzarello et al.’s analysis of subjects’ spontaneous development of dragging modalities led to the determination of a classification (Arzarello et al., 2002), which researchers have referred to as the “dragging schemes” (Olivero, 2002).

Abduction

In the previous section, the notion of abductive processes is mentioned. Peirce was the first to introduce the notion of abductive inference, and compare it with other inferences, such as deduction and induction. According to Peirce,

> “abduction looks at facts and looks for a theory to explain them, but it can only say a "might be", because it has a probabilistic nature. The general form of an abduction is: a fact A is observed; if C was true, then A would certainly be true; so, it is reasonable to assume C is true” (Peirce, 1960, p. 372).

Recently, researchers have renewed interest in abduction. In particular, Magnani defines abduction in a way that we find quite useful. According to him abduction is,

> “the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated” (Magnani, 2001, pp. 17-18).

Moreover, the following distinction of direct abduction versus creative abduction will be useful for our study. *Direct abduction* is when the “rule” used in the abductive process consists of a theorem that is already known to the student; while *creative abduction* is when the “rule” of the abduction consists of something new, that is not previously known by the student (see also Magnani, 2001; Thagard, 2006). Other researchers have studied various uses of abduction in mathematics education (Reid, 2003), and abductive processes in relation to transformational reasoning (Simon, 1996; Cifarelli, 1999; Ferrando, 2006). The basic idea is that an abductive inference may serve to organize, reorganize and transform problem solvers’ actions (Cifarelli, 1999). Abductive processes have also been observed by Arzarello et al. (1998) during the development of conjectures when students were using the dragging schemes, as mentioned above. In the next section we describe how our work builds on that of Arzarello et al., trying to study in detail the processes that occur during the conjecturing stage in open problem investigations, how these processes may be fostered, and what they might lead to during the phase of proof production.

OUR HYPOTHESES

While Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 2002) focused their attention on the subjects' use of the dragging schemes during the development of a conjecture, we concentrate on the abductive processes that may be induced by certain dragging schemes. Arzarello et al. observed that abduction occurs during solvers' use of the dragging schemes. Moreover, they claim that the production of conjectures is based on abductive processes. Thus, it seems that the use of certain dragging schemes may foster abductive processes, and, consequently, the production of conjectures. To some extent, the dragging schemes can be seen as cognitive artefacts (Norman,1991). We would like to investigate the relationship between the use of the dragging schemes and the development of abductive processes. In order to accomplish this investigation we need to induce solvers' use of dragging schemes, so we decided to introduce students to the specific dragging strategies. This way we seem to be able to induce the use of specific dragging schemes for the solution of open problems and, consequently, the appearance of abductive processes.

Below is a hypothesis of what might occur as a solver, who has been introduced to the dragging schemes, approaches an open problem in a DGE.

- Step 1: conscious use of different dragging strategies to investigate the situation – after *wandering dragging*, in particular *dummy locus dragging* (or *lieu muet dragging*) to maintain a geometrical property of the figure (*intentionally induced invariance*, or III), and use of the *trace tool*.
- Step 2: consciousness of the locus (*lieu*) that appears through *lieu muet dragging* – this marks a shift in control from ascending to descending – and description of a second invariance (*invariance observed during dragging*, or IOD).
- Step 3: hypothesis of a conditional link between the III and the IOD, to explain the situation.
- Other forms of dragging may be performed: *line dragging*, *linked dragging*, and the *dragging test*.
- Step 4: formulation of a conjecture of the form 'if IOD then III' (product of the abduction).
- Step 5: production of a mathematical proof of the conjecture (or attempt of it). Potential re-formulation of the conjecture.

The Notion of *Path* and an Example

Another hypothesis that we advance is that there is a key element, the *path*, that plays a fundamental role in the abductive process. In this section, we will try to introduce the concept of *path* and its significance for the model.

One of the dragging schemes, *lieu muet dragging*, involves dragging a point with the intention of maintaining a given property of the figure (which becomes the III). Some regularity may appear during this dragging stage, leading to the discovery of particular constraints that the dragged point has to respect (expressed in the IOD). Because of their origin from dragging, such constraints may be interpreted as the property of the point to belong to a particular figure. In mathematical terminology, that of course may not be consistent with students' way of thinking, we can speak of a hidden locus (*lieu muet*). Such locus can be made explicit by the trace tool, through which it appears on the screen (*lieu parlante*). During lieu muet dragging the solver notices regularities of the point's movement and conceptualizes them as leading to an explicit object. We refer to this object as a *path* when the solver gains consciousness of it, as generated through dragging, and consciousness of its property that if the dragged point is on it, a geometrical property of the Cabri figure is maintained. In this sense a *path* is the reification (Sfard, 1991) of a *lieu* that can now be used in a "descending control" mode (Arzarello et al., 2002). Zooming into Step 2, above, we observe that this is the point of the process in which the notion of *path* arises, and we can add a Step 2bis to indicate the (potential) geometric interpretation of the *path*, in order to (potentially, after Step 3) perform *line dragging*, *linked dragging*, and the *dragging test along such path*.

We believe that the *path* plays an important role in relation to the abductive processes that can be used to develop conjectures in a DGE. In particular, recognition of a *path* can act as a bridge, fostering the formulation of a conjecture. In fact, the *path* can be used during the abductive processes, but then it may no longer appear (or it may appear in a different form) in the formulation of the conjecture. Below, we zoom into a way in which abductive processes may take place and lead to a derived conjecture, and then we provide an example of the model in use during an activity.

- Intentionally Induced Invariance (III): the solver tries to maintain a certain geometrical property.
- Invariance Observed during Dragging (IOD): the solver notices that when he/she drags a certain basic point X along the *path*, the III seems to be maintained.
- Product of abductive process: it becomes reasonable for the solver to assume that if point X lies on the *path* (description of the IOD), the III is true.

If the path is recognized as a particular geometrical figure *F*, the derived conjecture may be: if X lies on F, the III is true.

Activity: Draw three points A, M, K, then construct point B as the symmetric image of A with respect to M, and point C as the symmetric image of A with respect to K.

Construct point D as the symmetric image of B with respect to K. Drag M and make conjectures about ABCD. Then try to prove your conjectures.

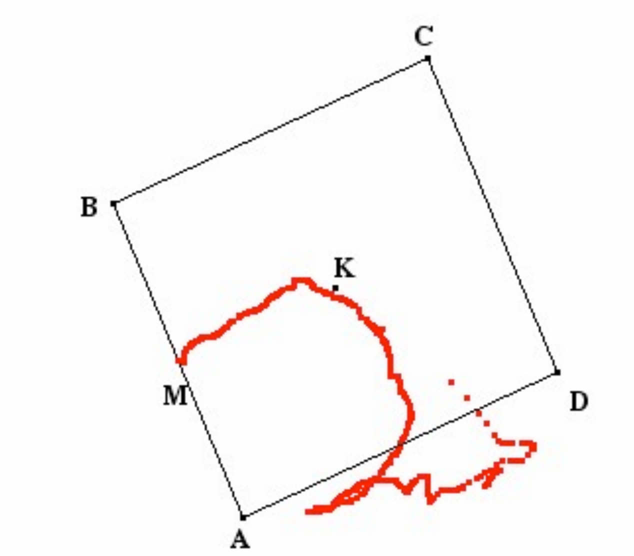

Fig 1: Dragging with the trace tool can help a student notice a *locus* (or *lieu*).

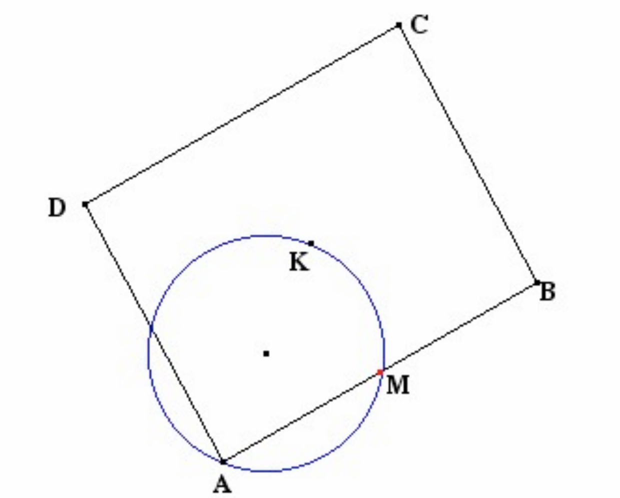

Fig 2: M is being dragged along the path (*line dragging*).

A Response [2]: Through *wandering dragging* solvers may notice that ABCD can become different types of parallelograms. In particular, they might notice that in some cases ABCD seems to be a rectangle (they can choose this as the III). With the intention of maintaining this property as an invariant, solvers might mark some configurations of M for which this seems to be true, and through the trace tool, try to drag maintaining the property, as shown in Fig 1. This can lead to noticing some regularity (IOD) in the movement of M, which might lead to awareness of an object along which to drag (the circle of diameter AK, potentially not yet recognized as "a circle"). At this point, when such awareness arises, we can speak of *path* with respect to the regularity of the movement of M.

If solvers recognize the path to be a familiar geometrical object, like in this case, they might be inclined to constructing it, as shown in Fig 2, and dragging along it (*line dragging*), or even linking the free point to it (*linked dragging*) and performing a *dragging test*. Through this abductive process, as an attempt at explaining the experienced situation, as Magnani describes (Magnani, 2001), solvers may hypothesize a conditional link between the III and IOD. At this point the abduction leads to a hypothesis of the form 'if IOD then III', leading to a conjecture like the following: "If M is on the circle of diameter AK, then ABCD is a rectangle," or (if they discover or derive a property of the base-points which is equivalent to M lying on the circle): "If AKM is a right triangle, ABCD is a rectangle."

In the case of the first conjecture, here is how we hypothesize the abduction (*creative abduction*) might go.

- III: ABCD is a rectangle.
- IOD: when M dragged along the *path*, fact A seems to be true. The path is a known geometric figure*:* the circle of diameter AK.
- Product of the abduction: If point M lies on the circle of diameter AK, ABCD is a rectangle.

This product of the abduction coincides with a formulation of a conjecture. However, solvers might also perform a second abduction (this time a *direct abduction*) linking the property "M belongs to the circle" to a property of the base-points of the construction. In this case this may lead to a formulation of the conjecture like: "If the triangle AMK is a right triangle (with ∠AMK as the right angle), ABCD is a rectangle." In this case the further elaboration of the geometrical properties recognized in the path will have led to a key idea (Raman, 2003) of a possible proof. In particular, this idea together with that of triangles AMK and ABC being similar, should be enough for students to successfully provide a proof to their conjecture. In this sense, abductive processes involving the notion of *path* (as a reified concept the solver is aware of) might be a step towards the achievement of cognitive unity [3] (Boero, Garuti, & Mariotti, 1996; Pedemonte, 2003).

Some Research Questions

Given the hypotheses outlined above, we propose some general questions for a research study. First, it would be interesting to investigate what forms of reasoning (abductive, deductive, ...) are actually used (and how) during the conjecturing stage of an open problem in a DGE. In particular, if subjects use *lieu muet* dragging, what is the role of the *path*? Can our model be confirmed (even in a potentially modified version)? Second, how does a DGE contribute to the development of the proof of a conjecture? It would be interesting to compare the dragging schemes (if any) used during this stage to those used during the conjecturing stage. It might also be insightful to investigate the forms of reasoning used during the conjecturing stage in the cases in which subjects do produce a proof. Finally, it would be interesting to study whether it is possible to detect a relationship between the forms of reasoning analyzed, and, if possible, to describe such a relationship.

EXPERIMENTAL DESIGN AND POTENTIAL CONCLUSIONS

We propose to structure the study in the following general way: by a selection of the subjects, the introduction of the subjects to the dragging schemes, finally open-problem-activity-based interviews on pairs of students. We will use results from the pilot study to refine the model, the research questions, and the activities proposed during the interviews. In the results of this study we hope to be able to include: a description of some cognitive processes that occur during the conjecturing stage of the investigation of open problems in a DGE; and validation of the model (or of a revised version of it), or motivations for rejecting it as a useful descriptive model. Therefore, this study should help gain better comprehension of specific cognitive processes. In particular, we hope to gain some insight into how abductive processes may occur, whether they can be fostered by preliminary introduction of the dragging schemes, and how the notion of *path* may foster the formulation of conjectures.

A secondary objective is to gain insight into how a DGE contributes to the development of proofs. The activities proposed during the interviews will all be open problems in which students are asked to make conjectures and then try to prove them. The *path* might also play a role in the generation of a proof, in that it may be a part of the "reorganization and transformation" that occurs with abductive reasoning (Cifarelli, 1999). This might very well be new powerful tool for the solver to use in a potential proof (or solution of the problem) as an aid to gain cognitive unity, as mentioned above. In this case, it would be reasonable to hypothesize that if the dragging schemes were to foster abductive processes, and abductive processes were to foster cognitive unity, then introducing the tool of the dragging schemes to the students a priori might accelerate and facilitate the entire process of making a conjecture and reaching a proof for it.

If our hypotheses are confirmed, and the dragging schemes and the notion of *path* do contribute positively to the formulation of conjectures (and potentially of proofs), we will recognize them as tools of semiotic mediation, with a semiotic potential that could be exploited by teachers. In this case, teaching experiments, which introduce the dragging schemes at a class-level, should be carried out, in order to further investigate how the teacher can exploit the semiotic potential of the dragging schemes in the classroom practice. Later, large-scale quantitative research on the induction of cognitive processes through introduction of the dragging schemes could be conducted, with the didactic objective of implementing the teaching of the dragging schemes in school curricula.

NOTES

1. With "static" and "dynamic" referred to conjecture, here we intend to emphasize the nature of the conjecture's origin.

2. This is only one of the many possible responses leading to this specific conjecture. Of course different students might reach this conjecture in different ways. Moreover there are many different conjectures that students can formulate by focusing their attention on different geometric invariants (in this case, having ABCD be a kite, a rhombus, or a square).

3. Boero et al. introduce cognitive unity as follows: "During the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingled with the justification of the plausibility of his/her choices: during the subsequent proving stage, the student links up with his process in a coherent way, organizing some of the justifications ("arguments") produced during the construction of the statement according to a logical chain" (Boero, Garuti, & Mariotti, 1996, p.113).

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STUDENT JUSTIFICATIONS IN HIGH SCHOOL MATHEMATICS

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In this paper, we continue our previous work on evaluating the use of structured derivations in the mathematics classroom. We have studied student justifications in 132 exam solutions and described the types of justifications found. We also discuss the results in the light of frameworks for types of understanding and reasoning that have been previously presented in research.

Keywords: student justifications, structured derivations, high school, instrumental and relational understanding, imitative and creative reasoning

INTRODUCTION

The ability to justify a step in, for instance, a proof can be considered a skill that needs to be mastered, at least to some extent, before proof is introduced. In a wider sense, proof can even be regarded as justification (Ball and Bass, 2003). Unfortunately, students are not used to justify their solutions (Dreyfus, 1999). It is common for teachers to ask students to explain their reasoning only when they have made an error; the need to justify correctly solved problems is usually de-emphasized (Glass & Maher, 2004). Consequently, without the explanations, the reasoning that drives the solution forward remains implicit (Dreyfus, 1999; Leron, 1983).

A previous study (Mannila & Wallin, 2008) indicated that high school students can improve their justification skills in one single course. In this paper, we will present the results from a follow-up study, focusing on the types of justifications given by the students. We will first discuss some related work and also give a brief introduction to the approach used when teaching the course. The main research questions are as follows:

- What types of justifications do students give in a solution?
- Do the types of justifications change as the course progresses?

RELATED WORK

Justifications as a condition for proof

The importance of proof and formal reasoning for the development of mathematical understanding is also recognized by the National Council of Teaching Mathematics (NCTM), which issues recommendations for school mathematics at different levels. According to the current document (NCTM, 2008), students at all levels should, for instance, be able to communicate their mathematical thinking, analyze the thinking of others, use mathematical language to express ideas precisely, and develop and evaluate mathematical arguments and proof. While discussing mathematical ideas is important, communicating mathematical thinking in writing can be even more efficient for developing understanding (Albert, 2000).

To think mathematically, students must learn how to justify their results; to explain why they think they are correct, and to convince their teacher and fellow students. "[M]athematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. Readers who can only decode words can hardly be said to know how to read. … Likewise, merely being able to operate mathematically does not assure being able to do and use mathematics in useful ways." (Ball & Bass, 2003; p. 29)

Justifications are not only important to the student, but also to the teacher, as the explanations (not the final answer) make it possible for the teacher to study the growth of mathematical understanding. Using arguments such as "Because my teacher said so" or "I can see it" is insufficient to reveal their reasoning (Dreyfus, 1999). A brief answer such as "26/65=2/5" does not tell the reader anything about the student's understanding. What if he or she has "seen" that this is the result after simply removing the number six (6)?

Types of understanding and reasoning

A review of literature on mathematics education shows that there is an interest in studying the distinction between being able to apply a determined set of instruction in order to solve a mathematical problem and being able to explain the solution by basing it on mathematical foundations.

Skemp (1976) discusses two types of understanding named by Mellin-Olsen: *relational* ("knowing both what to do and why") and *instrumental* ("knowing how", "rules without reasons"). People who exhibit an instrumental understanding know how to use a given rule and may think they understand when they really do not. For instance, getting the correct result when applying a given formula is an example of instrumental, not relational, understanding. One typical example can be found in equation solving, where students learn to "move terms to the other side and change the sign", without necessarily knowing why they do it.

More recently, Lithner (2008) has created a research framework for different types of mathematical reasoning. He distinguishes between two main types: *imitative* and *creative*. Imitative reasoning is rote learnt and can be divided into two subtypes: memorised reasoning, where the student, for instance, solves a problem by recalling a full answer given in the text book or by the teacher, and algorithmic reasoning, where a problem is solved by recalling and applying a given algorithm. The other main type, creative reasoning, includes a novel reasoning sequence, which can be justified and is based on mathematical foundations. One of the main differences between imitative and creative reasoning is that the former does not necessarily involve analytical and conceptual thinking, whereas such thinking processes are essential to creative reasoning.

STRUCTURED DERIVATIONS

The course was taught using structured derivations, which can be seen as a logic-based approach to teaching mathematics (Back & von Wright, 1998; Back & von Wright, 1999; Back et al, 2008a). The format is a further development of Dijkstra's calculational proof style, where Back and von Wright have added a mechanism for doing subderivations and for handling assumptions in proofs. Using this approach, each step in a solution/proof is explicitly justified.

In the following, we illustrate the format by briefly discussing an example where we want to prove that $x^2 > x$ when $x > 1$.

•	Prove that $x^2 > x$:	*task*
-	$x > 1$	*assumption*
||-	$x^2 > x$	*term*
≡	{ Add $-x$ to both sides }	*justification*
	$x^2 - x > 0$	*term*
≡	{ Factorize }	…
	$x(x - 1) > 0$	
≡	{ Both x and x-1 are positive according to assumption. Therefore their product is also positive. }	
	T	

The derivation starts with a description of the task ("Prove that $x^2 > x$"), followed by a list of assumptions (here we have only one: $x > 1$). The turnstile (||-) indicates the beginning of the derivation and is followed by the start term ($x^2 > x$). In this example, the solution is reached by reducing the original term step by step. Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is transformed to the second one.

Another key feature of this format is the possibility to present derivations at different levels of detail using subderivations, but as these are not the focus of this paper, we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the articles by Back et al.

A feedback analysis (Back et al., 2008b) showed that students highly appreciated that they had to justify each step of their solutions. Student feedback also indicated that students appreciate that solutions become easier to understand both during construction and afterwards, but also when reading others' solutions.

Why use in education?

As each step in the solution is justified, the final product contains a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983).

The explicated thinking facilitates reading and debugging both for students and teachers.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that has commonly been the result of teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format also has the potential to function as mental support, giving students belief in their own skills to solve the problem. Also, as solutions and proofs look the same way using structured derivations, the traditional "fear" of proof might be eased. Furthermore, the use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different detail levels.

STUDY SETTINGS

Data collection

The data were collected during an elective advanced mathematics course on logic and number theory (about 30 hours) that was taught at two high schools in Turku, Finland, during fall 2007. All in all, twenty-two (22) students completed the course at either school and participated in the study (32 % girls, 68 % boys). The students were on their final study year.

The course included three exams (E1, E2, E3) held after 1/3, 2/3 and at the end of the course. The exams were of increasing difficulty level, i.e. E1 was the easiest and E3 the most difficult one. Two assignments from each were chosen for the analysis. We have thus analysed the justifications in six solutions for all 22 students, that is, 132 solutions in total.

Method

The data collected, i.e. the justifications, were of qualitative nature. Qualitative data are highly descriptive, and in order to interpret the information, the data need to be reduced. In this study, a content-analytical approach was chosen for this purpose. The basic idea of content analysis is to take texts and analyze, reduce and summarize them using emergent themes. These themes can then be quantified, and as such, content analysis is suitable for transforming textual material into a form, which can be statistically analyzed (Cohen, 2007).

A first round of the content analysis was done by one of the authors, who analyzed 18 solutions from E1 and 24 solutions from E2. This initial coding resulted in a first view of the types of justifications produced by the students. The authors discussed the results and agreed on how to combine the detailed justifications into higher-level categories. Next, all solutions were analyzed using the preliminary categories as the coding scheme. The second round analysis showed that the categories found in the initial phase were sufficient for covering all justifications found in the 132 solutions.

A quantitative approach was then taken in order to be able to do statistical analysis of the data and to illustrate the results graphically.

The use of both quantitative and qualitative methods has several benefits. Mixed methods avoid any potential bias originating from using one single method, as each method has its strengths and weaknesses. A mixed methods approach also allows the researcher to analyze and describe the same phenomenon from different perspectives and exploring diverse research questions. Whereas questions looking to describe a phenomenon (``How/What..?", our first research question) are best answered using a qualitative approach, quantitative methods are better at addressing more factual questions (``Do...", our second research question) (Cohen, 2007).

RESULTS AND DISCUSSION

In this section, we will present and discuss the results from our study. We will arrange the presentation around the research questions and the related literature put forward in the beginning of the paper.

Types of justifications

The content analysis revealed six main justification types:

- *Assumption*: Use of assumption: input given assumptions or rewrite the assumption and then use it.
- *Vague/broad statement*: A very brief and uninformative justification type: "logic" or "simplify".
- *Own explanation*: An explanation for *why* the step is valid in own words and/or with symbols, e.g. "$2k^2+2k$ is an integer if k is an integer. Therefore $2(2k^2+2k)$ is an even integer". In some justifications a mathematical definition was written out in own words, e.g. "$2 \equiv_{13} 106$ because $2 - 106 = -104$, $13 \mid -104$".
- *Procedural description:* An explanation of *what* is done in the step, i.e. a verb. E.g. "add 2x+4 to both sides", "add -3 to both sides" and "calculate the sum".
- *Rule*: A name of a rule or a definition. E.g. rule for absolute values, the conjugate rule, tautology, congruence etc. In some cases, the justification also included the rule explicitly written our in text.

Figure 1 illustrates the proportion of different justification types found in the three exams respectively. Naturally, the type of justification chosen in a certain situation is closely related to the assignment and/or the step at hand. For example, assumptions or rules will not be used in problems where there are no assumptions or rules to apply.

In problems that involve assumptions, it is common for the assumption to be used only once or twice. Thus, it is natural that the proportion of this type of justifications is quite low. However, when looking at how students managed to use assumptions in solutions, the analysis showed that all students but one were able to handle assumptions correctly already in the first exam, i.e. after 1/3 of the course.

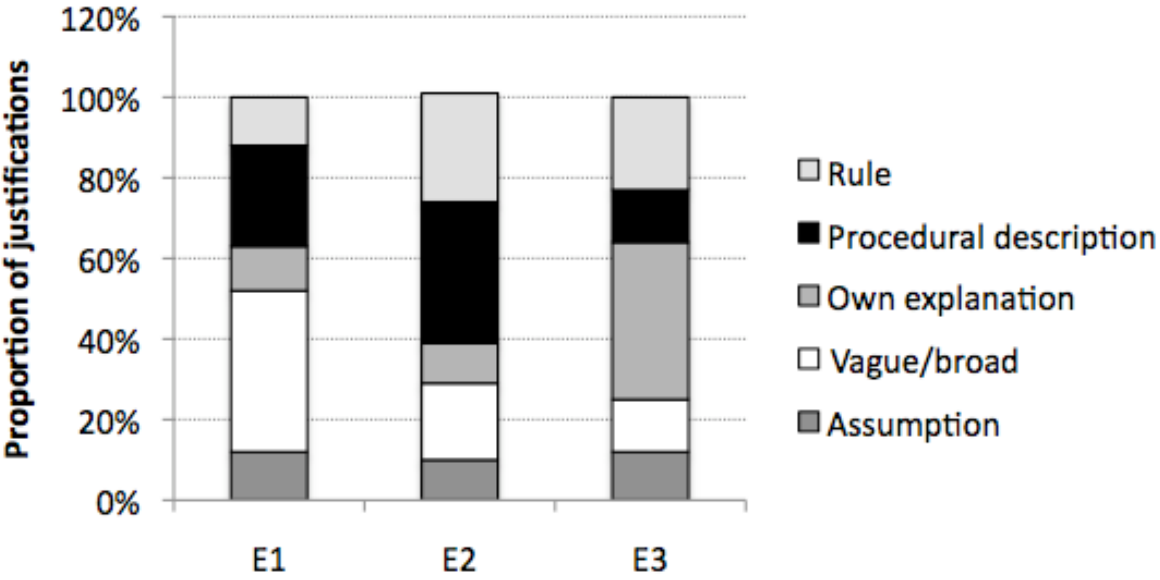

Figure 1: The proportion of justifications of different types in the three exams.

The proportion of procedural descriptions and rule applications varied. The first exam involved equations, and as a result the proportion of procedural descriptions (e.g. “add -3 to both sides”) was quite large. No specific rules were necessarily needed to solve these assignments, giving a rather low proportion for rule justifications. The second exam dealt with more advanced equations and logic, whereby the importance of rules increased. In the final exam, the main topic was number theory, and here we see a large increase in own explanations, whereas the proportion of procedural descriptions and rule justifications decreases.

When students gave a rule as a justification, most usually stated only the name of the rule, whereas only a few also wrote out the rule. In the final and most difficult assignment, where the rule was central to the solution, a larger proportion of students (46 %) had written out the rules explicitly, compared to those who had only written the name of the rule (22%).

The most noticeable changes are found for “vague/broad” justifications and “own explanations”. As the frequency of the former decreases, the number of the latter increases. This may have several explanations. It is reasonable to assume that students justify difficult assignments more carefully. Consequently, the occurrence of own explanations increase. The first exam was naturally easier than the last one: as easy assignments include more “straightforward” steps, students may not have seen the need to justify those steps in any more detail. This is supported by the results from our feedback study, where students found “extra writing” unnecessary for simple tasks (Back et al., 2008b).

Instrumental and relational understanding

Only two types of justifications, “own explanations” and “procedural descriptions”, involved students writing in their own words. There is an important difference between the two types. In a "procedural description", students wrote *what* they did, but not why they had chosen or were allowed to do so. The "own explanation", on the other hand, also gave information regarding *why* the step was valid.

This is closely related to Skemp's instrumental and relational understanding (Skemp, 1976). Own explanations are clearly relational, but the procedural descriptions cannot easily be mapped to either type of understanding. Although Skemp argued that justifications such as "move -3 to the other side" are examples of an instrumental approach to understanding, we do not think the situation is as black-or-white.

We feel that the procedural descriptions do not reveal enough information about whether the student has truly understood what he or she has done. For instance, in the course on which this study is based, the teacher used the addition rule when solving equations (i.e. "add -3 to both sides "). How can we tell if a student who later writes the shorter "move -3 to the other side" has *understood* instrumentally or relationally? In our opinion, ruling out the possibility of relational understanding in such situations requires more than the mere justification.

To exemplify this, we now look at three different solutions to an assignment involving absolute values. The absolute value rule referred to below is the following:

$$|x| = c \Leftrightarrow c \geq 0 \wedge (x = c \vee x = -c)$$

- **Kim:** *procedural description, relational understanding*

  Kim did not use the rule for absolute values learnt in class, but rewrote the expression in a way showing that he had really understood the absolute value concept. The solution was correct and indicated a relational understanding of the concept absolute value.

  $|x - 4| = 2x - 1$

  $\Leftrightarrow$ { rewrite the absolute value }

  $(x - 4 = 2x - 1 \wedge x \geq 4) \vee (-x + 4 = 2x - 1 \wedge x < 4)$

- **Layla:** *procedural description, instrumental or relational understanding*

  Layla used the absolute value rule and solved the problem correctly. Despite the correct solution, we cannot know whether Layla understood the concept or merely used a rule she had learnt that "should work" for this type of problems.

  $|x - 4| = 2x - 1$

  $\Leftrightarrow$ { using rule for absolute values}

  $(x - 4 = 2x - 1 \vee x - 4 = -2x + 1) \wedge 2x - 1 \geq 0$

- **Joe:** *procedural description, instrumental understanding*

  Just like Layla, Joe also justified the initial step with "using rule for absolute values". However, he used the rule incorrectly, as he "forgot" the second part of it (the requirement on x).

$|x - 4| = 2x – 1$

⇔ { using rule for absolute values}

$x – 4 = 2x – 1 \lor x – 4 = -2x + 1$

This was, however, a rather common error (made by almost 36% of all students in one of the assignments in E1). Had Joe and the other students had a relational understanding for absolute values, the additional requirement would have been clear to them even if they had forgotten what the rule looked like.

Thus, it seems as if one can in fact conclude that a given procedural justification is *not* an example of relational understanding – this is the case if the step is incorrect. However, doing the opposite, i.e. concluding that a procedural justification to a correct step is relational, is in our opinion quite difficult. Even the thinking process behind a “simple” justification, such as “simplify”, may have been complex.

Is a clearly relational approach always needed?

In high school mathematics, much time is spent on things like solving equations and simplifying expressions. Thus, to a large extent it boils down to using rules, and consequently a seemingly instrumental approach becomes dominant. However, this is foregone by the teacher explaining the theory behind the rules and the definitions. If the student later uses the rules in an instrumental or a relational way is up to how well he or she understood the theory. If the underlying concept is not clear to the students, the rules are most likely applied without reasons, i.e. instrumentally. One area of high school mathematics where relational understanding most likely becomes more evident is in textual problems, where students first need to formalize the problem specification. In order to correctly specify the problem, the student needs to understand the problem domain and the underlying concepts. Relational understanding is naturally also important when constructing proofs.

Furthermore, sometimes a justification with a seemingly instrumental approach is the best one that can be given. Take for example a complex trigonometric expression. Finnish high school students have a collection of rules that they can always have with them, even on exams. One can hardly require them to start explaining rules in order to be allowed to apply them. What is essential in such a situation is that they a) have an underlying understanding for trigonometry, b) know how to apply trigonometric rules correctly, and c) they are able to manipulate the expression into a form where one of the many rules can be applied correctly.

As another example we can take equation solving and the “add -3 to both sides” type of procedural justification mentioned above. Let us say we have two students: one who understands that whenever you have an expression of the form $a = b$, you can add the same value to both sides without changing the truth value of the full expression ($a + c = b + c$), and another who knows that one should move “lonely numbers” to the other side while changing the sign. Both of these students would probably use similar justifications, but only one of them would have an instrumental understanding. The one who understands relationally would most certainly not write out the rule ($a = b \Leftrightarrow a + c = b + c$), which would be needed in order for the teacher to be able to distinguish the justification from that given by the other student.

Imitiative reasoning

According to Lithner (2008), students are usually able to imitate a previously seen example when solving an exercise. However, in this study students justified their solutions in differing ways. The teacher went through an example in class, which was similar to one of the problems in the first exam. When comparing the example to the students' solutions, we found only a few similarities, suggesting that students had found their own ways of justifying instead of memorizing the teacher's explanations. One could argue that memorizing the entire solution becomes more tedious using structured derivations than in traditional mathematics: it is not as easy memorizing both the procedure and the justifications as it is to memorize only the procedure.

Justifications and validity of steps

As was described above, a "correct" justification can lead to an incorrect derivation step. This can happen for several reasons, one being the one exhibited by Joe above: not completely remembering a rule.

Careless mistakes in a step do not seem to correlate with the type or the accuracy of the justification. Only a small number of this type of errors was found (in 9% of the assignments throughout all three exams), which was also supported by students' feedback as they pointed out that they made fewer careless mistakes using structured derivations than what they usually do (Back et al., 2008b).

CONCLUDING REMARKS

The study presented in this paper is a continuation on earlier qualitative studies on the use of structured derivations in education. Previous results indicate that students appreciate the approach ("it takes me longer, but I understand better") and that it improves students' justification skills already during one single course (Mannila & Wallin, 2008). Furthermore, we have found that explicit justifications make students think more carefully when solving a problem (Back et al., 2008b). With this study, we now also have a rather clear picture of how students motivate their solutions and how these change throughout the course.

Getting students to clearly document their solutions step by step is a step forward, although "judging" the justifications is everything but trivial. Thus, many questions still remain. Is it possible to teach a way of writing "good" justifications? And if we want to try, what characterizes such justifications? This makes it difficult to "judge" student justifications.

Another aspect, which we have not considered so far, is related to teachers and course books. How do teachers justify their solutions when teaching using structured derivations? How are examples justified in texts? In order for students to develop relational understanding, we believe that it is essential that examples are explained freely (”using own words”) as often as possible.

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VISUAL PROOFS: AN EXPERIMENT

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The main goal of this paper is to start a preliminary study on the basic features of visual proofs in mathematics and their use in mathematics teaching. The investigation, based on college mathematics students, shows a very poor use of visual reasoning in the performance of mathematical tasks. Moreover, the use of visual semiotic systems is not natural in students but needs some special training. Some students' ways of working with visual proofs have been identified, showing that most often diagrams are not seen as representations of complete processes, but rather as ready-made aids to solve problems.

INTRODUCTION

Many researchers have stressed the importance of visual reasoning in the learning of mathematics and have remarked that research in mathematics education has still a lot to develop about this topic (see e.g. Dreyfus 1991, Jones 1998, Presmeg 2006). In this perspective this paper focuses on visual proofs i.e. on proofs given by figures, diagrams or graphs only, possibly equipped with numbers, letters, arrows, dots, or other signs. Although we are taking into account geometrical figures only, we will use the expression 'diagrammatic proof' or 'visual proof' in a more inclusive sense. Working with visual proofs involves constructing and treating (detaching, reversing, superposing, translating,…) geometrical figures and extracting information from them. All these operations will make evident the inferential steps that make up a visual proof of a statement.

The main goal of this paper is to take into account some features of diagrammatic proofs that might be relevant from the educational viewpoint and to explore the opportunities that diagrammatic proofs can provide in order to improve the approach to mathematical theorems. Diagrams are not relevant just as far as visual proofs are concerned, but they can play a role also to support both standard proof processes (i.e., proofs based on a verbal text) or problem solving.

Lately interest in visual proofs has grown up mainly in the fields of mathematical logic and mathematics education. On the side of mathematical logic above all we mention the work of Barwise and Etchemendy (1991) and further developments on the same line such as Jamnik's book (2001). From the educational viewpoint the role of visualization had been undervalued by people arguing that geometrical thinking could be reduced to algebraic one. More recently the role of visual reasoning in mathematics education has been taken again into account and emphasized (see e.g. Dreyfus 1991, Dvora & Dreyfus 2004, Hanna 1989, Presmeg 1997, 2006). At this regard, a number of works, such as Nelsen's books (1993, 2001) have provided a wide selection of examples of visual proofs from different sources.

THEORETICAL FRAMEWORK

We did not find in literature a theoretical framework closely focused on visual proofs. Although here we are focusing on visual proofs that are based on geometrical figures, we take into account some different works about visual reasoning and visualization that could help us to interpret difficulties about this topic. First of all, according to Fischbein (1993), geometrical figures are mental entities (named also 'figural concepts') which possess conceptual and figural characters at the same time. In this frame, as other studies in geometry, we refer to figures as the mental entities which possess properties imposed by, or derived from axiomatic systems and to drawings as their (external) representations. A major problem in the use of diagrams and figures are the potential conflicts between conceptual and perceptual features of figures. Fischbein's theory is very helpful at this regard. Fischbein argues that '…figural concepts constitute only the ideal limit of a process of fusion and integration between the logical and figural facets' (Fischbein 1993, p.150). In particular in visual proofs there are logical facets concerning the deductive method based on diagrams and figures. In fact one has to consider that visual proofs are bound to correspond to some extent to proofs in the standard mathematical sense. In this work I do not mean to question the rigorousness of diagrammatic proofs (on this topic see Barwise & Etchemendy (1991), Jamnik (2001), Hanna & Sidoli (2007), Allwein, G. & Barwise J. – Eds. (1996) and references therein) but I assume that they can be regarded as legitimate mathematical processes.

Another main difficulty encountered by students is due to the lack of coordination of systems of semiotic representations (Duval 1993). Working with a visual proof requires a continuous interplay between the semiotic system of figures and the semiotic systems involved in the statement, usually verbal texts or symbolic expressions. Like Duval, I assume that semiotic systems are not neutral carriers of meanings but can contribute to the construction of meaning themselves. This explains the attention I am going to pay to semiotic systems through this paper.

In this context we propose to answer to the following research questions:

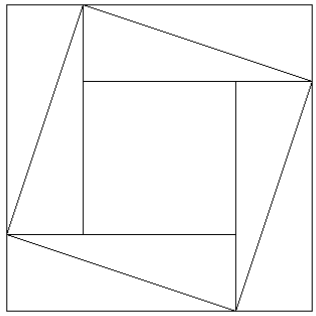
- What are the main difficulties in understanding visual proofs?
- To which extent previous knowledge (mathematical notions and techniques) can influence the comprehension of diagrammatic proofs?

AN EXPERIMENT

At the Università del Piemonte Orientale, in Italy, in the context of a course devoted to mathematical proof, we have given a group of 13 sophomore and third year undergraduate Mathematics students a number of tasks requiring to look at diagrammatic proof of some statement and to reconstruct such a proof (i.e. to describe how the proof could be extracted by the figure.) The tasks have been administered as written tests and they were followed by interviews in order to better understand the arguments written by students.

The problems are the following:

Task 1.

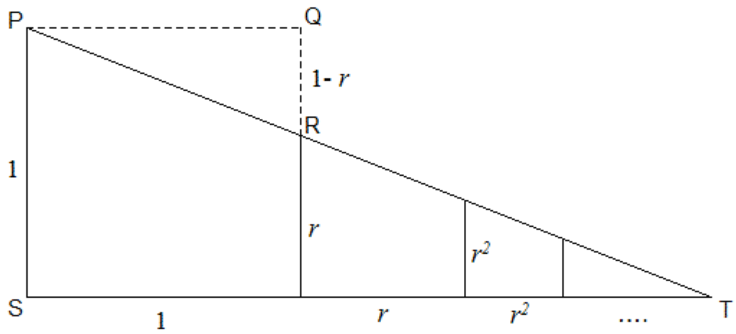


The picture on the right represents a visual proof of the Pythagoras' theorem.

- Describe such a proof.
- Reconstruct the figure in the case that the legs of the right-angled triangle have the same measure.

Task 2.

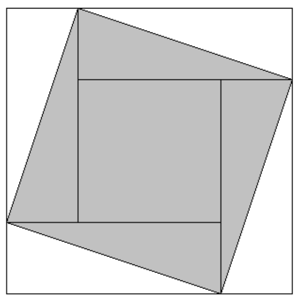
The following picture represents a visual proof of the theorem $\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r}$ for $0 < r < 1$.



Describe such a proof.

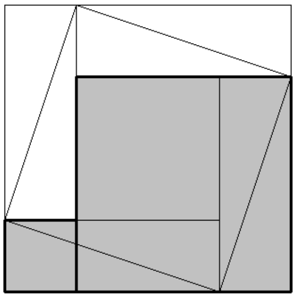
We chose two statements in two different fields of mathematics: the Pythagoras' theorem and the geometric series. Pythagoras' theorem is customarily associated to visual representations, whereas the latter is less common (at least in Italy), as the convergence of the geometrical series is usually proven using a combination of algebraic and analytical arguments. So this visual proof is very unusual for Italian mathematics college students.. The choice of theorem from different fields is aimed at finding common features and common difficulties.

Analysis of task 1.



This problem is mainly based on visual arguments. We think that students could meet with difficulties in the identification of the areas of squares whose sides are the legs of the right triangle. Such a problem is related to the rearrangement of the figure. In particular students have to recognize that the area on the hypotenuse is constructed by subtracting from the whole figure the area of the four white triangles, as shown in the picture on the right,

then noticing that one can get the same area by subtracting from the whole figure four other triangles as follows



and finally remarking that this area corresponds to the sum of the areas of the squares whose sides are the two legs of the original triangle. We think that students could meet with difficulties from both the perceptual side, as they might fail to spot the appropriate triangles or square, and the lack of coordination between the figural system and the verbal one.

Analysis of task 2.

In order to perform this task students could divide the resolution of the problem into three steps. Notice that in addressing the problem these three steps are not necessarily separated, but on the contrary they can influence each other in order to understand the visual proof and students could not be aware of this subdivision. The steps are:

- *Matching labels with the formula.*

This requires just an one-way transcription of parts of the formula ($\sum_{i=0}^{+\infty} r^i$, $1-r$, 1) to a visual representation on a plane (sides of triangles i.e. segments of measure $\sum_{i=0}^{+\infty} r^i$, $1-r$, 1). Its success mainly depends on labels, for example indicating measure of segments, on other graphical signs such as ellipses and so on. Also this can present some difficulties. One has to identify $\sum_{i=0}^{+\infty} r^i$ with $\overline{ST}$. This could be quite simple if one understands the meaning of the dots '…'. More difficult is to understand a correspondence between $\frac{1}{1-r}$ and the figure. Some students could recognize $1-r$ as he length of $\overline{QR}$ and perhaps 1 as the length of $\overline{PS}$ or $\overline{PQ}$.

- *Reconstruction of the drawing.*

In order to perform the task successfully, students have to reconstruct the drawing as a process. In this case the construction should have a well-fixed order: first, one draws the right-angled trapezoid with two equivalent sides of measure 1 and the shorter parallel side of length r; second, one draws the right-angled trapezoid with two equivalent sides of measure r and the shorter parallel side of length r^2; third, one draws the right-angled trapezoid with two equivalent sides of measure r^2 and the shorter parallel side of length r^3, and so on. While doing this one has to check that the respective oblique sides have all the same slope so that the trapezoids sum up in a triangle with vertex T. This could be done using for example similitude of triangles, or comparing ratio of sides, or writing down the equation of the straight-line. It seems reasonable that the success in this topic should depend on the knowledge of the mathematical notions above. In this case students should not just know the necessary ideas, but also be ready to exploit them in an effective way.

- *Main step.*

Students have to understand that the relation $\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r}$ comes out from a proportion $\frac{\sum_{i=0}^{+\infty} r^i}{1} = \frac{1}{1-r}$ i.e. $\left(\sum_{i=0}^{+\infty} r^i\right):1 = 1:(1-r)$ and that this could be related with the proportion of the measure of segments in figure. The treatment of the formula could be difficult in itself, but the coordination between the formula and the figure might prove harder. Then students have to understand that the triangles PST and PQR are similar to identify the corresponding sides, and to write down the appropriate proportion i.e. $\overline{ST}:\overline{PQ} = \overline{PS}:\overline{QR}$, and so $\left(\sum_{i=0}^{+\infty} r^i\right):1 = 1:(1-r)$.

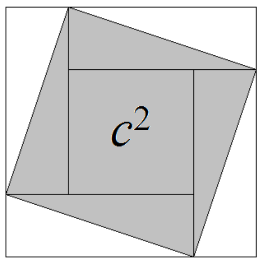
RESULTS

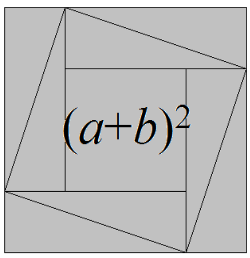
Task 1.

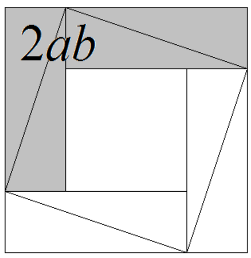
First of all in this task few students only provided an explicit description of the construction process of the figure**.** In this visual proof, the construction of the drawing is not related to the understanding of the proof since they succeed to achieve it with deductive arguments not based upon the whole picture but on some parts of it only. Second, all of them introduced letters a, b, c to indicate the measure of the sides of the triangle in order to find correspondence between the formula $a^2+b^2=c^2$ and the figure. Finally, students addressed the first task in three different but not necessarily separate ways:

1. *Modifying the formula in order to find correspondence with the figure*

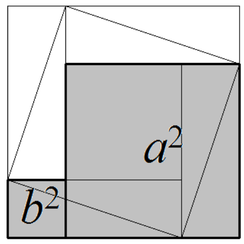
Some students tried to connect the formula $a^2+b^2=c^2$ to the figure and to identify just c^2 in the picture to the right. They were not able to do the same for a^2 and b^2. Then they wrote down $(a+b)^2-2ab=c^2$ most likely because they could find $(a+b)^2$ and $2ab$ in the picture too, as shown below:







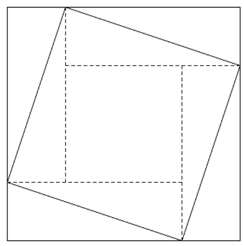
This way the students recognized the remaining area a^2+b^2 in the figure on the right. This kind of proof is mostly based on visual arguments except for the initial modification of the formula.



2. *Area computation*

This strategy is the most common in problems of this kind. It consists in calculating the area of the external figure in two different ways and then comparing the results to obtain the required relationship.

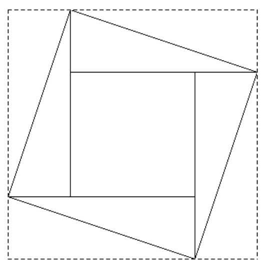
In the first problem they calculate the area of the square of side $a+b$ i.e. $(a+b)^2$ and then the same area as the sums of the five subfigures (four triangles and a square of side c) i.e. $4\frac{ab}{2}+c^2$. Comparing the two expressions they got the Pythagorean theorem through algebra. In this case they did not consider the dashed lines in the picture on the right.



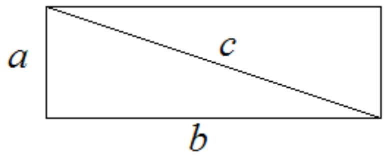
3. *Figures as plain tools*

The figure is not seen as a process embodying the proof of a statement but just as a tool that can be used to occasionally pick some piece of information useful to get a proof.

For example in this problem four students considered just the tilted square of side c and its five subfigures (four right-angled triangles and the square of side a-b). Actually they did not consider the dashed lines in the figure on the right. Comparing the area of the square of side c calculated as c^2 with the same area but regarded as the sum of the areas of the five subfigures one obtains the result as in point *2 (Area computation).*



Another student just considered the rectangle



defining a the short side and b the long one. Then she used a so called “circular argument” or “begging the premise” (cf. Weston, 2000), i.e. she used the Pythagoras’ theorem to get $c=\sqrt{a^2+b^2}$ and hence squaring both sides she got the Pythagoras’ theorem $c^2=a^2+b^2$.

Notice that also the answers in point *2. (Area computation)* denote that the figure is not seen as an autonomous process of proof.

Task 2.

The Problem 2 proved the most difficult one. Nobody succeeded in understanding this visual proof. So a hint was given to them while they were solving the task. I told them that a fundamental tool for its comprehension was the similitude of triangles and in particular the proportionalities between corresponding sides of the triangles. After that some of them succeeded to recognize that Δ PST and Δ PQR are similar and they found the correspondence between the formula and the sides of triangles.

As a first result we have that students were able to match labels with the formula, and to understand the meaning of the dots '…'. As second finding we have that most students did not reconstruct the drawing. The reasons are three:

1. Students understood the need to reconstruct the drawing. Such construction is a necessary step in order to consider the visual proof as a process. Unfortunately they are not able to do such a reconstruction. One can see this outcome from the following excerpts:

   A: Consider a square of side of length 1 ($l = r^0$) PQMS and construct a right-angled triangle PST such that the shorter leg is $\overline{PS} = r^0$ and one finds that the longer leg $\overline{ST}$ is the sum of infinite segments having measure respectively r^0, r^1, r^2, …..

   (Student A understands that the measure of $\overline{ST}$ is not an assumption but a finding of the construction but he can not prove that result, as it became clear from the interview) or

   B: I can not understand how in the figure r^2 comes out from r.

2. Students consider figures just as plain tools. This is evident in task 2:

   C: …from figure I can see that $\overline{PS}$ measures 1, $\overline{ST}$ measures $\sum_{i=0}^{+\infty} r^i$,…

   Student C do not see that $\overline{PS} = 1$ is an assumption while $\overline{ST} = \sum_{i=0}^{+\infty} r^i$ is a deduction and in particular it means that the series converges.

3. Students understand the need to reconstruct the drawing but they do not do it since they considered it trivial.

Finally we have that students can conclude the proof using the help given to them, but we distinguish

- students who are able to prove that the triangles PST and PQR are similar because they remember this notion
- students who do not remember this notion or never learnt it

In this case the problem is that even if students know well similitude of triangles they do not think to introduce such "new" tool which could not be directly deduced with a simple manipulation of the objects which already appear in the proof.

General discussion

One of the main findings of this work is that visual proofs are not seen as processes but the figures are just plain tools which help to find results. The investigation of the protocols highlights that the unsuccessful results of this kind of task are due not only to the semiotic system of figures or to the conflict between the conceptual and figural nature of visual proofs but it comes out that the concept of mathematical proof is not understood enough. But it comes out above all from the fact that they do not feel the necessity of reconstructing the drawing Moreover, in the first problem students use just some parts of the figure and not the whole of it. Also this behaviour, in some cases, is due to a misunderstanding of the nature of process of visual proofs. The role of graphical signs and more in general of the perceptive learning of a figure is very important both in a positive and in a negative sense i.e. it can be either a help or a hindrance in understanding visual proofs as pointed out by Duval (1993). Moreover some students did not attribute values at all to graphical signs as explained in the analysis of the first task.

Second it comes out that one of the main obstacles is the lack of geometrical knowledge: notions like similitude and congruence of triangles, Thales' theorem, etc. are hardly known, which severely prevents any attempt to work with the figure. This situation is found in the problem about the geometrical series. For example one of the fundamental steps for understanding this visual proof is to notice that the triangles Δ PQR and Δ PST are similar. No one spotted this geometrical fact. There might be two reasons of it. First, students have never learnt this or they have forgotten it. Second, they do not have this notion "ready to use", actually they to some extent know the notion of similitude of triangles but they are not used to work with it. The fact is that Italian students work very little or do not work at all about the visualization of geometrical figures (for further details see Mariotti, 1998). Moreover, the time given for solving the task is not sufficient to remember or to reconstruct this notion. However, the necessity to use tools and constructions which are not directly related to objects at hand is a common feature in mathematical proofs, which do not belong to visual proofs only. Students could not overcome the difficulty of introducing such new elements in the visual proof we proposed them. Such difficulty, however, is not peculiar to visual proofs only. Indeed we do not find this problem in the first task. In this case all student succeed to grasp the result even if in an incorrect way, for example using the figure just as a tool to extract information. Here one has just to manipulate the formula of the Pythagorean theorem or manipulate its figure, there is no need to introduce new constructions, techniques, assumptions, tools, etc.

Finally the analyses of protocols show that students prefer to work with algebra instead of using visual arguments coming from manipulation of figures. The visual proof in task one is only of visual nature but no one addressed it with just visual arguments. Just one student used prevalently visual arguments (see strategy 1), but even in this case there was a preliminary modification of the formula.

CONCLUSION

This explorative research outlined the lack in visual reasoning by a group of Italian mathematical college students. This lack is due to different reasons: little knowledge of certain basic mathematical tools, poor acquaintance with the use of figural representations, conflict between the conceptual and figural nature of diagrammatic proofs and sometimes poor understanding of the concept of mathematical proof. In this context tasks like those presented in this work could help students to develop some important tools to approach other mathematical problems such as problem solving and standard proof processing. Anyway they point out that it is very difficult to learn proof without some basic pieces of mathematical knowledge.

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MODES OF ARGUMENT REPRESENTATION FOR PROVING – THE CASE OF GENERAL PROOF

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In light of recent reform recommendations, teachers are expected to turn proofs and proving into an ongoing component of their classroom practice. At least two questions emerge from this requirement. Is the mathematical knowledge of high school teachers sufficient to prove various kinds of statements? And does their knowledge allow the teachers to determine the validity of an argument made by their students? The results of the present study point to a positive answer to the first question in the framework of elementary number theory (ENT). However, the picture is much less positive with respect to the second one.

THEORETICAL BACKGROUND

The calls for enhancing students' abilities to prove and to refute mathematical statements appear prominently in various reform documents of different countries (e.g., Israeli Ministry of Education, 1994; National Council of Teachers of Mathematics [NCTM], 2000). In the NCTM document, reasoning and proof is one of five process standards for all grade levels. Still, there is a need to clarify what proof is in the classroom context. Stylianides (2007) made an attempt in this direction:

> Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> - It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justifications;
> - It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
> - It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 107).

Stylianides' (2007) definition talks about the classroom community as the authority to determine the correctness of a proof. However, the teacher, as the representative of the mathematics community, has a special role in the endeavor. He needs to be attentive to both – the mode of argument for a

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given statement (such as general proof, counter example, supportive example), as well as the mode of argument representation (such as numerical, verbal or symbolic), to be able to determine the correctness of a justification.

To what extent are teachers prepared to implement proofs and proving as part of their classroom practice? Relatively little is known on teachers' subject matter knowledge in this area. Dreyfus (2000), following Healy and Hoyles' (1998) work with high school students, presented 44 secondary school teachers with nine justifications to the universal claim "The sum of any two even numbers is even". He found that most secondary school teachers easily recognized formal proofs, but had little or no appreciation for other types of justifications such as verbal, visual or generic ones. Knuth's (2002b) findings suggest that secondary school teachers recognized the variety of roles that proofs play in mathematics. Noticeably absent, however, was a view of proofs as tools for learning mathematics. Many of the teachers held limited views of the nature of proof in mathematics and demonstrated inadequate understandings of what constitutes proofs.

In a different study on in-service high school teachers' knowledge of elementary number theory (ENT), only a third of the 36 teachers provided counter examples to the (false) universal statement "All commutative actions are also associative" (Zaslavsky & Peled, 1996).

These studies focused solely either on universal or on existential statements. Tirosh (2002) presented the same group of elementary and middle school teachers with both universal and existential ENT statements. Tirosh and Vinner (2004) analyzed 38 prospective middle-school teachers' written answers to questionnaires on the issues of constructing and evaluating proofs and refutations in ENT. They found that about 20% of the prospective teachers incorrectly argued that some of the existence theorems in the questionnaires are false (e.g., "There exists a real number b so that $a + b < a$"). Furthermore, about half of the prospective middle school teachers incorrectly argued that numerical examples that satisfy existential statements are just examples and could not be regarded as mathematical proofs. These responses suggest that some prospective teachers develop a general view that a mathematical statement is true only if it holds for "all cases", a view which is adequate for universal statements but not for existential ones.

The present study addresses a high school teacher's knowledge with respect to universal and existential statements in the area of ENT. It aims to give a preliminary answer to the following two questions. Is the mathematical knowledge of high school teachers sufficient to prove ENT statements? And does their knowledge allow the teachers to determine the validity of an argument made by their students?

Note: the work of Tirosh (and Vinner) and Dreyfus differ from the work presented here in the population and the mathematical statements.

METHOD

Participants

A group of 50 high school teachers participated in the research. All teachers had some experience teaching in high school. Ms R was one of the teachers. Ms R was chosen as focus teacher for this study on the basis of her answers to the set of questionnaires below.

When participating in our project, Ms R had been teaching for five years in a high school, working with high-achieving students from a high socio-economic background. In parallel, she was studying for her Master's degree in mathematics education. The program included a number of mathematics courses and a number of psycho-didactical courses.

Tools

In one of these courses, the participants' mathematical knowledge was analyzed through their written reactions to two questionnaires that dealt with six ENT statements. No time-limit was imposed for the work on the questionnaires. In this section we briefly describe each of the questionnaires.

Predicate Quantifier	Always true	Sometimes true	Never true
Universal	**S1.** The sum of any five consecutive natural numbers is divisible by 5. *True/General proof*	**S2.** The sum of any three consecutive natural numbers is divisible by 6. *False/Counter example*	**S3.** The sum of any four consecutive natural numbers is divisible by 4. *False/Counter example*
Existential	**S4.** There exists a sum of five consecutive natural numbers that is divisible by 5. *True/Supportive example*	**S5.** There exists a sum of three consecutive natural numbers that is divisible by 6. *True/Supportive example*	**S6.** There exists a sum of four consecutive natural numbers that is divisible by 4. *False/General proof*

Table 1. Classification of the six statements

The Prove-Questionnaire was intended to identify the participants' production of proofs (validations and refutations) to various (true or false) statements. The questionnaire included six ENT statements (statements S1-S6 in Table 1). The statements were chosen to include one of three predicates (always true, sometimes true or never true), and one of two quantifiers (universal or existential). Clearly, the validity of a statement is determined by the combination of its predicate and its quantifier. Three of the statements are true (S1, S4, S5), and the other three are false (S2, S3, S6). Table 1 displays the six statements according to their quantifier and predicate; their truth value as well as a suitable proof method are also indicated. The participants were asked to examine each of the statements, to determine whether it is true or false, and to prove their claim.

The *True or False-Questionnaire* was intended to check the participants' identification of the correctness of 43 justifications for the six statements they had proven before, between six and nine justifications for each statement, using numerical, verbal or symbolic modes of arguments representations. For each justification, the participants were asked to determine whether it verifies (refutes) the statement, and to explain their evaluation. The justifications were presented as if they were written by students in various modes of argument representations.

In analyzing teachers' answers to the first and second questionnaire we related to the modes of argumentations as well as to the mode of argument representations.

RESULTS AND DISCUSSION

In this section we first present the participants' answers to the *Prove-Questionnaire*, with examples of Ms R's proofs. Then we discuss the participants' answers to the *True or False-Questionnaire.* Here we narrow the discussion to five justifications which relate to two statements – S1 and S6. We chose these two statements because they require general proofs. We present in detail the answers of Ms R to each justification, followed by a brief description of the results for all participants with regard to the same justifications.

Prove-Questionnaire

All the teachers produced correct proofs to each of the six statements. That is, the modes of argumentation the teachers chose for each statement were appropriate. Their proofs were presented in one of two modes of argument representation – symbolic or numeric (see Table 2).

All participants used the symbolic mode of argument representation for statements S1 and S6, which required a general mode of argumentation. About half of the participants produced numerical examples to refute the universal statements S2 and S3, and the majority of the participant provided a single numerical example to validate the existential statements S4 and S5. None of the participants provided several examples to prove or refute a statement. These findings indicate that the participants who used numerical examples knew when an example is sufficient for proving a statement.

	S1	S2	S3	S4	S5	S6
Numeric	---	50	44	72	80	---
Symbolic	100	50	56	28	20	100

Table 2: Percentages of modes of argument representation produced by the participants (N=50)

We present Ms R's proof for statement S1 which is a universal, always true:

> Let's denote five consecutive numbers by a, $a+1$, $a+2$, $a+3$, $a+4$. Their sum is: $a+a+1+a+2+a+3+a+4 = 5a+10$.
> $(5a+10):5 = a+2$. $a+2$ is a natural number for any a that is a natural number. Therefore the statement is true.

As we can see, the proof that Ms R provided related to all the cases in the domain, used correct inference rules, is concise, and thus exemplifies a sound proof.

Ms R's proof for statement S6, an existential, never true statement shows similar characteristics:

> Let's check: a is a natural number. $(a+a+1+a+2+a+3):4=(4a+6):4$
> We divide the last expression by 2, obtaining $(2a+3):2$. But, $2a+3$ is an odd number (the sum of even, $2a$ and odd, 3), and therefore is not divisible by 2. The statement is not true.

Again Ms R correctly identified the need for a general mode of argumentation, and used a symbolic mode of argument representation.

True or False-Questionnaire – Ms R's explanations.

We now focus on the two statements that required general proofs, meaning that the general mode of argumentation should be used. Yet, such an argument can be displayed in at least two modes of argument representation – verbal and symbolic. Five sets of justifications, Ms R's judgments, and her explanations are presented. A short discussion follows each set.

Example 1: Verbal justification to statement S1 and Ms R's explanation

The given correct justification:

> Moshe claimed: I checked the sum of the first five consecutive numbers: *1+2+3+4+5=15* is divisible by 5. The sum of the next five consecutive numbers is larger by 5 than this sum (each number is bigger by 1 and therefore the sum is bigger by 5), and therefore this sum is also divisible by 5. And so on, each time we add 5 to a sum that is divisible by 5, and therefore we always obtain sums that are divisible by 5. Therefore the statement is true.

Ms R's judgment: Moshe's argument is not correct.

Ms R's explanation

> Moshe checked the case *1+2+3+4+5=15*, which can be accidentally true. In proving one needs to generalize, and therefore Moshe's justification is not correct.

From Ms R's explanation we can learn that she correctly identified the mode of argumentation needed for proving S1. Yet, she failed to notice the coverage aspect in Moshe's justification.

Example 2: Verbal justification to statement S1 and Ms R's explanation

The given correct justification

> Mali claimed: I first tried the first ten examples of 5 consecutive numbers:
>
> 1+2+3+4+5=15 2+3+4+5+6=20 3+4+5+6+7=25
> 4+5+6+7+8=30 5+6+7+8+9=35 6+7+8+9+10=40
> 7+8+9+10+11=45 8+9+10+11+12=50
> 9+10+11+12+13=55 10+11+12+13+14=60.
>
> I saw that the statement is true for the first ten. All other sums of five consecutive numbers are obtained by adding multiples of 10 to one of the listed sums (for instance, the sum 44+45+46+47+48 is obtained by adding multiples of 10, 5 times 40, to the sequence: 4+5+6+7+8 that I checked before). Since multiples of 10 are also divisible by 5, the statement is true.

Ms R's judgment: Mali's argument is not correct.

Ms R's explanation

> Here also there is no generalization to all the natural numbers, and therefore this is incorrect. It is not a proof.

From Ms R's explanation in this case we can learn that Ms R is concerned with the mode of argumentation. She did not identify the cover aspect in Mali's correct verbal justification.

Example 3: Symbolic justification to statement S1 and Ms R's explanation

The given incorrect justification

> Ayala claimed: Among any five consecutive numbers, there is one that is divisible by 5. Let's look at a sequence of five consecutive numbers : $5x$, $5x+1$, $5x+2$, $5x+3$, $5x+4$ ($5x$ is divisible by 5). The sum of this sequence is: $5x+(5x+1)+(5x+2)+(5x+3)+(5x+4)= 25x+10$, and $25x+10$ is divisible by 5 for any x. Therefore the statement is true.

Ms R's judgment: Ayala's argument is correct.

Ms R's explanation

> x represents any number, and therefore the proof is general.

Ms R's explanation in this case relates to two important observations. x represents any number, and in this sense the justification is general. However, $5x$ represents a multiple of five, and thus the sequence 1, 2, 3, 4, 5, for instance, is not included. Hence, Ayala's justification is correct for only a subset of the cases that one needs to relate to in order to prove S1. Ms R failed to notice this flaw in Ayala's justification.

Example 4: Verbal justification to statement S6 and Ms R's explanation

The given correct justification

> Moshe claimed: I checked the sum of the first four consecutive numbers: $1+2+3+4=10$, ten is not divisible by 4. The sum of the next four consecutive numbers is obtained by adding 4 to this sum (each of the four numbers in the sum grows by 1, so the sum grows by 4). It is known that adding 4 to a sum that is not divisible by 4 will yield a sum that is not divisible by 4 either. And so on, each time we add 4 to a sum that is not divisible by 4, and therefore we always obtain sums that are not divisible by 4. Therefore the statement is not true.

Ms R's judgment: Moshe's argument is not correct.

Ms R's explanation

> Moshe chose an example, and on the basis of this example he concluded that there are no such four numbers. But maybe if he would have picked up four other numbers it could have been correct.

Once more, Ms R's reaction exemplifies her view that Moshe's verbal explanation is an example. Again she correctly determined that for this statement an example is not an appropriate mode of argumentation.

Example 5: symbolic justification to statement S6 and Ms R's explanation

The given incorrect justification

> Ayala claimed: Among any four consecutive numbers, there is one that is divisible by 4. Let's look at a sequence of four consecutive numbers : $4x$, $4x+1$, $4x+2$, $4x+3$ ($4x$ is divisible by 4). The sum of this sequence is: $4x+(4x+1)+(4x+2)+(4x+3) = 16x+6$. $16x$ is divisible by 4 for any x, while 6 is not divisible by 4. So, the sum $16x+6$ is not divisible by 4. Therefore the statement is not true.

Ms R's judgment: Ayala's argument is correct.

Ms R's explanation

> Ayala proved the claim for all four numbers, and hence it is not possible to show that there are four numbers, hence the justification is correct.

The same phenomenon as in example 3 is evident again in Ms R's reaction. On the one hand, it shows that she fully understands the mode of argumentation needed, but on the other hand she fails to recognize whether the given justification carries the general aspect needed.

It seems that for Ms R, the symbolic mode of argument representation, assures that the cover aspect of the proof is taken care of. Also, for Ms R, a verbal mode of argument representation is judged to be merely a numerical example.

One may wonder whether Ms R is unique in her judgments. Let's return to the entire population of 50 participants and check how many teachers made similar choices as Ms R.

For the first statement (S1), 34 percent of the participants rejected the correct verbal justifications (Examples 1 and 2), on the ground that they are not general, and at the same time accepted the incorrect symbolic justification (Example 3), on the ground that it is general. As Ms R, these teachers correctly identified the mode of argumentation needed for each statement.

For the last statement (S6), 26 percent of the participants rejected the correct verbal justification (Example 4), on the ground that it is not general, and at the same time accepted the incorrect symbolic justification (Example 5), on the ground that it is general. Also in this case, the teachers correctly identified the mode of argumentation needed for each statement.

Twenty percent of the participants were consistent in their answers, that is made the same choices as Ms R in the cases of the five justifications presented above.

SUMMING UP AND LOOKING AHEAD

The present study addressed the following two questions. Is the mathematical knowledge of high school teachers sufficient to prove mathematical statements from the field of elementary number theory? And does their knowledge allow the teachers to determine the validity of an argument made by their students?

Our findings indicate that the participants were able to produce correct proofs and refutations to the statements presented. While the teachers chose correct modes of argumentation for each statement, it was evident that they were concerned with this aspect in the second questionnaire.

The picture emerging from the *True or False-questionnaire* seems more complex. About a third of the teachers failed to identify as universal the general-cover aspects of the given arguments in verbal modes of representation. These findings substantiate similar findings reported by Dreyfus (2000), that teachers tend to perceive verbal proofs as deficient because they lack symbolic notations. However, Dreyfus (2000) found that teacher tended to reject verbal justifications. Our findings indicate that teachers had difficulties in understanding verbal justifications, but they did not reject them as such. Teachers' difficulties with verbal justifications are particularly worrying in light of the results reported by Healy & Hoyles (2000), namely that high school students not only preferred verbal proofs due to their explanatory power but also that their verbal arguments were more often deductively correct than their arguments in other modes of representation, yet at the same time they expected to get low grades for such proofs.

A quarter of the participants failed to identify when symbolic justifications did not cover all cases in the domain. These findings substantiate findings reported by Knuth (2002b): "In determining the argument's validity, these teachers seemed to focus solely on the correctness of the algebraic manipulations rather than on the mathematical validity of the argument" (p. 392). When being presented with an algebraic justification, the teachers' focus was on the examination of each step, ignoring the need to evaluate the validity of the argument as a whole.

The everyday practice of teachers involves a constant evaluation of students' justifications for statements. It is likely that verbal or symbolic justifications of the kinds presented in our study, will emerge during interactions with students. Therefore, it is important that teachers will be familiar with verbal justifications and able to judge their validity.

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ARGUMENTATION AND PROOF: A DISCUSSION ABOUT TOULMIN'S AND DUVAL'S MODELS

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In this paper, we discuss the idea of a gap between argumentation and proof, an idea we think to be prevailing in the educational institution. Our claim is that the only use of propositional calculus is insufficient to the analysis of the validation process in mathematics and could artificially reinforce that idea of a gap. This claim can be understood as a criticism of Toulmin's and Duval's model, a criticism we hope to be a constructive one. We are then brought to the following proposal: taking explicitly into account the logical quantification and the mathematical objects in the models could help to explain mathematical creativity.

INTRODUCTION: THE PREGNANT IDEA OF A GAP

The issue at stake in this paper is the relationship between argumentation and proof. It seems to us that the assumption of a gap prevails in the educational institution. This prevalence could have major effects on mathematic education:

> « Is it possible, yes or no, to shift from one to the other without too many efforts or misunderstandings?
>
> […]
>
> If one answers No, one admits there is a gap between the cognitive processes of argumentation and the deductive reasoning at stake in a proof: the use of argumentation could not but maintain or even reinforce the obstacles and misunderstandings about what a proof is, because its discursive process acts against a valid reasoning process in ordinary language. » (Duval, 1992, p. 43, our translation)

Willing to take into account this gap between argumentation and proof, which is theorised in Duval's works, part of the teachers have been induced to put forward specifically the formal aspect of the proof (through structuring attempts like "I know that", "Now", Therefore" for example) and to distinguish this aspect from the work on the content of statements. This phenomenon can be seen in Kouki's thesis (2008) through a survey carried out among six Tunisian teachers about learning and teaching of equations, inequalities and functions. Moreover, Kouki shows, through a more extended experimental study (involving 143 pupils and students in their transient period between secondary school and higher education) the consequences of these theoretical conceptions on the students' practices which tend to apply formal procedures as much as possible. In another context, Segal (2000) highlights the tendency of UK students to evaluate proof validity only from their formal aspect. There are a lot of examples of this phenomenon. We shall focus on two specific ones in order to point out the stakes of this issue.

Example 1

This example is taken from Barrier (2008) in which an extract of Battie (2003) is analysed. In this paper, a group of three students in scientific upper sixth form are asked to evaluate the following statement $\forall a \forall b (GCD(a,b) = 1) \Rightarrow (GCD(a^2, b^2) = 1)$. The group starts an argumentation built on the choice of some coprime natural numbers (3 and 2, 2 and 5, 9 and 17 then 4 and 15) and on the evaluation of the GCD of their respective square. Here is an extract of their dialogue (translated from French).

1. A :	Or 125 and 16. They are relatively prime.
2.	I don't know.
(Laughes)	
3.	You set 125 divided by 16 and you'll see… No, it is not the right way to do it. 16 by 16 is 4 2, 2 times 2/
4. A :	No, I think 16 and 125 are relatively prime.
5.	Yeah, when we square the things/
6. A :	Yeah, but we don't know, it's not written in the text book, but we can't prove it in the general case /
7.	Oh we make fun of it!
8. A1 :	We can't use it then. Well, I think, I really don't know, the teacher may have told it.

In (3), a student undertakes a prime factorization of 16. This method could be used for the emergence of a proof of the analysed statement. However, it seems to us that the students, influenced by their school culture regarding proof, disregard this possibility. They act as if the evaluation of a statement through an argumentation built on the manipulation of objects and the search for proof were two distinct and independent activities.

Example 2

Alcock & Weber (2005) analyse how thirteen student volunteers taken from first-term, first year introductory real analysis courses check the validity of the following proof (they were asked to determine whether or not the proof was a valid one):

> Theorem. $\sqrt{n} \to \infty$ as $n \to \infty$
>
> Proof. We know that $a < b \Rightarrow a^m < b^m$.
>
> So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$.
>
> $n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n.
>
> So $\sqrt{n} \to \infty$ as $n \to \infty$ as required.

The inference between the two last propositions is invalid. Exactly two students rejected the proof because they had familiar counter-examples. Their rejection was not founded on the recognition of a logical gap between the propositions. Three other students rejected the proof. They did it because they failed to recognize what they thought to be a proof structure. In particular, they argued that the definitions of the mathematical concepts involved in the argument were not used. Their decision seems to be grounded on exclusive formal considerations. From the point of view of mathematical activity, this is a misconception: definitions are not always employed in a mathematical proof and, above all, very few mathematical proofs are enough detailed so that their logical structure can be recognized without any work. To finish with this example, notice that while only two students refused the proof because of an invalid warrant, ten did it when the interviewer helped them to interpret "$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n" as "the series is increasing" and "$\sqrt{n} \to \infty$ as $n \to \infty$" as "the series is divergent". Our hypothesis is that this last intervention allowed the students to enter the semantic content of the proposition. Precisely, the translation into ordinary language could help them to go to a semantic interpretation in a familiar domain in which they know that there is some increasing and convergent series.

We shall now undertake a criticism of Duval and Toulmin's models which are often used in research in mathematical didactics about argumentation and proof (Mathé (2006), Tanguay (2005), Inglis & al. (2007), Pedemonte (2007, 2008)). Our main thesis is that using the proposition (in the sense of propositional calculus, as opposed to predicate calculus) as a basic element of modelling leads to overestimate the gap between argumentation and proof. In particular, we consider that taking into account mathematical objects and quantification in the didactical analysis allows a quite different approach to the validation process in mathematics.

BRIEF PRESENTATION OF DUVAL AND TOULMIN'S MODELS

We shall begin with a brief presentation of Duval's approach. Let us use Balacheff's presentation (2008, p. 509):

> ''Deductive reasoning holds two characteristics, which oppose it to argumentation. First, it is based on the operational value of statements and not on their epistemic value (the belief which may be attached to them). Second, the development of a deductive reasoning relies on the possibility of chaining the elementary deductive steps, whereas argumentation relies on the reinterpretation or the accumulation of arguments from different points of view. (Following Duval 1991, esp. p. 240–241).''

Duval often stands out that only argumentation lies on the content of propositions whereas what is important in a proof is the operating status of the proposition (in other words the way the proposition fits into the formal structure of the "modus ponens").

> "This brings a first important difference between deductive reasoning and argumentative reasoning. The latter appeals to implicit rules which depend partly on the language

structure and partly on interlocutors' representations: therefore the semantic content of the propositions is essential. On the contrary, in a deductive step, the propositions do not intervene directly according to their **content** but according to their **operating status**, that is to the position previously assigned to them in the step process" (Duval, 1991, p. 235, our translation)

Duval especially focuses on this argument to support the idea that proof and argumentation involve very different cognitive processes. In this matter, Balacheff (2008, p. 509) points out that:

> « One can imagine how this should raise question in our field considering that other researchers give a central role to ‘‘mathematical arguments’’ and ‘‘mathematical argumentation’’ in their consideration of what proof is.”

Recently, Toulmin's model has been used in many works focused on reasoning from a mathematics education viewpoint. The following example shows how Pedemonte (2008, p. 387) presents Toulmin's restricted model:

> “In Toulmin’s model an argument consists of three elements (Toulmin, 1993):
>
> C (claim): the statement of the speaker.
>
> D (data): data justifying claim C.
>
> W (warrant): the inference rule, which allows data to be connected to the claim.
>
> In any argument the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin’s terminology the standpoint is called the claim. The second step consists of the production of data supporting the claim. The warrant provides the justification for using the data conceived as a support for the data-claim relationships. The warrant, which can be expressed as a principle, or a rule, acts as a bridge between the data and the claim.”

This model has been used to analyse as well the production of arguments as the production of proof. In particular, Pedemonte uses this model to compare argumentation and proof relationships. Therefore the three elements (C, D, W) must be considered as more inclusive than the ternary structure of "modus ponens" (A, $A \rightarrow B$, B) used by Duval to analyse the proof in the sense where the Toulmin's model warrant is not necessarily a theorem. Nevertheless, these two models share a common point by both using the proposition in the sense of propositional calculus as a basic element of modelling. Mathematical objects and quantification are not explicitly taken into account in the model structure.

AN EXAMPLE OF USE OF A QUANTIFICATION THEORY

Several attempts have been made to use first-order theories in order to help analysing mathematical reasoning in our research team (natural deduction in Durand-Guerrier & Arsac (2005) and Durand-Guerrier (2005), Tarski's semantics in Durand-Guerrier (2008), Lorenzen's dialogic logic and Hintikka's game semantics in Barrier (2008)). The ambition of these theories is to allow for the relationships between the semantic and syntactic aspects to be taken into account in the validation activities. On the contrary, Duval identifies a reasoning step when applying the "modus ponens" rule. He asserts for example:

> «The deductive step process is well known. It is defined by the fundamental rule "modus ponens", also called Law of Detachment." (Duval, 1992, p. 43, our translation)

We also saw that Toulmin's model rested on the same type of ternary structure. Durand-Guerrier & Arsac (2005, p. 151-152) showed that this standpoint was insufficient for analysing proof, especially in the case of analysis. Furthermore, the only "modus ponens" rule cannot exhaust the propositional calculus insofar as other deductive rules are necessary (Vernant, 2006, Chapter 3). Nevertheless, the deductive step derived from the Law of Detachment prevails in proof learning at lower secondary school and certainly deserves special attention. Our contribution will rather focus on the theoretical effects of this restriction: we consider that restricting the model to the propositional calculus induces to overestimate the distinction between argumentation and proof. Let us consider how Duval (1992, p. 44-45, our translation) analyses the following text by Sartre:

« Jessica :	Hugo ! You speak reluctantly. I watched you when you talked with Hoerderer :
	0. He convinced you.
Hugo :	1. No, he didn't convince me.
	2. Nobody can convince me that (one must lie to its friends).
	3a. But if he had convinced me.
	3b. It would be a reason more to shoot him.
	4. Because it would prove that he would convince other guys. »

Duval asserts that this argumentation appeals to the following deductive step:

Premise: If he had convinced me

Warrant: Nobody can convince me that one must…

Conclusion: (it would prove that) he would convince other guys.

This modelling leads Duval to draw the fundamental differences between argumentation and proof. Indeed, the argumentation step as modelled by Duval is quite different from the proof step based on the " modus ponens". Our questioning on this model induces us to suggest an alternative interpretation of this argumentation step based on natural deduction (Durand-Guerrier & Arsac (2005)). We note that xCy is the assertion that « x has convinced y ». The first step of Hugo's reasoning may then be interpreted in the following way:

Data:	$\forall x \neg (xCHugo)$ (2)
Inference rule:	universal instantiation
Conclusion:	$\neg (HoedererCHugo)$ (1)

We shall go on with the analysis of the reasoning (setting apart the assertion (3b) and identifying (4) with «he would convince other guys », i.e. removing what seems to refer to metalanguage) in the following way:

Data :	$HoedererCHugo$ (3a)
Inference rule:	existential generalisation
Conclusion:	$\exists x HoedererCx$
Data:	$\exists x HoedererCx$ (recycling)
	$\exists x HoedererCx \rightarrow \exists x \exists y (x \neq y) \wedge HoedererCx \wedge HodererCy$ *
	(implicit axiom)
Inference rule:	modus ponens
Conclusion:	$\exists x \exists y (x \neq y) \wedge HoedererCx \wedge HodererCy$ (4)

One shall notice that without the implicit axiom (*) (if Hoederer is able to convince one person, then he is able to convince two persons at least) the deduction from $HoedererCHugo$ to $\exists x \exists y (x \neq y) \wedge HoedererCx \wedge HodererCy$ would be invalid. Therefore it is necessary, in a way, to complete the reasoning to make it valid. In this extract, one does not know whether the implicit theorem applied is part of a set of statements which are jointly accepted by Hugo and Jessica. However, this type of completion is not exclusive to argumentation, since in mathematics a fully explained proof would be much too long and therefore illegible. Weber (2008) puts forward an experimental study on how proofs are checked by mathematicians. This does not mean that the check is limited to the good practices of inference rules: proof checking, including validation, calls on not only a search for sub-proofs but also for informal or example-based arguments.

Now, an important question to be raised is the relationship between proof and proposition content. In the analysed example, we used an implicit axiom to complete the formal analysis of reasoning. This axiom is linked to a certain idea we have about the interpretation field objects (human beings in this example), what Duval calls the semantic content of propositions. In particular, the implicit axiom (*) is based on the idea that human beings are more or less homogeneous. The purpose of the following paragraph is to show that the content of propositions also intervenes in the proof construction.

« CONTENT » OF PROPOSITIONS AND PROOFS

We use here an experiment from Inglis & al. (2007). Andrew, an advanced mathematics student, is confronted with the conjecture « if n is a perfect, then kn is abundant, for any $k \in IN$ ». Notice that a perfect number is an integer n whose divisors add up to exactly $2n$ and that an abundant number is an integer n whose divisors add up to more than $2n$.

ANDREW : Ok, so if n is perfect, then kn is abundant, for any k. Ok, so what does it, yeah it looks, so what does it mean ? Yeah so if n is perfect, and I take any p_i which divides this n, then afterwards the sum of these $p_i s$ is $2n$. This is the definition. Yeah, ok, so actually we take kn, then obviously all kp_i divide kn, actually, we sum these and we get $2kn$. Plus, we've got also, for example, we've also got k dividing this, dividing kn. So we need to add this. As far, as basically, there is no disquiet, k would be the same as this. Yeah. And, how would this one go ? [LONG PAUSE]

INTERVIEWER : So we've got the same problem as up here but in general ? With a … ?

ANDREW : Yeah. Umm, can we find one? Right, so I don't know. Some example.

INTERVIEWER : I've got some examples for you.

ANDREW : You've got examples of some perfect numbers ? OK, so 12, we've got 1 + 2 + 3 + 4 + 6, then, ok, + 12. [MUTTERS] But this is not ? Ok, perfect, I wanted perfect numbers. OK, so let's say six. Yaeh, and we've got divisors 2, 4, 6, 12. Plus I claim we've got also divisors. Yeah actually it's simple because, err, because err, the argument is that we've also got 1 which is divisor, and this divisor is no longer here is we multiply.

At the beginning of the interview, Andrew manipulates the definition of the concepts involved in the conjecture but this strategy fails to construct a proof. Then, he asks for examples and begins to play a semantic game which involves several numbers. As Duval says, this game increases the belief of the students in the validity of the conjecture (the epistemic value of the conjecture). In this sense, those kind of games are cumulative. However that argumentation which is linked with the content of the conjecture seems to be the clue of the completion of Andrew's strategy in his former attempt of proof construction. This is the manipulation of the perfect number 6 which provides to Andrew the idea that for all n, 1 is a divisor of $2n$ which is not equal to any 2k (with k a divisor of n). Pedemonte (2008) provides several convergent examples concerning algebra. In particular, she stands for the need of an argumentation which would integrate what she calls abduction steps in the proof construction process. In our example, the purpose is to explain why 12 would be abundant, starting from the fact that it is abundant (this practice is sometimes called the *analysis* of *analysis/synthesis* dyade). The proof approach (the synthesis of the dyade) is based on this explanation (12 is abundant because 1 is a divisor of 12 which is not a double of any divisor of 6). Pedemonte (2007, p. 32-33) also gives an example of this type of approach in geometry. Besides, from experiments carried out in set theory and analysis, Weber & Alcock (2004) underline the weakness of syntactic proof procedures ("unwrap definitions" and "push symbols") compared with semantic procedures (which call on object instantiations).

ABOUT TOULMIN'S COMPLETE MODEL

In their above-mentioned paper, Inglis & al. (2007) advocate the use of Toulmin's complete model which includes three new categories: the backing, the modal qualifier and the rebuttal which are introduced as follows (p. 4) :

> « The warrant is supported by the *backing* (B) which presents further evidence. *The modal qualifier* (Q) qualifies the conclusion by expressing degrees of confidence; and the *rebuttal* (R) potentially refutes the conclusion by stating the conditions under which it would not hold. »

The authors show that there are various types of *warrant* that the students (five students prepare a doctorate degree and one a master degree) connect with various *modal qualifiers*. They advocate the importance of inductive and intuitive justifications in the mathematical activity provided that these justifications are paired with the appropriate modal qualifier for the conclusion of the argument. They underline the interest for didactics researchers to use modal qualifiers specifically in the analysis process.

> "The restricted form of Toulmin's (1958) scheme used by earlier researchers to model mathematical argumentation constrains us to think only in terms of arguments with absolute conclusion." (Inglis & al., 2007, p. 17)

In his *remark on Toulmin*, Jahnke (2008, p. 370) makes this argument his own and emphasises the role of open general statements in mathematics. It seems to us that the role assigned to modal qualifiers in Toulmin's model shows that it is very difficult, in the didactic of mathematics, to integrate mathematical objects and their manipulation into models which are basically built from the propositional logic and from a syntactic approach of the mathematical activity.

CONCLUSION

Barrier (2008) advocates the necessity to appeal to transactional and intra-world procedures (Vernant, 2007) in order to explain mathematical creativity, i.e. to take into account the students' specific interactions with mathematical objects and the following decisions. The quantification theories, in particular the theories which develop a semantic point of view, allow to explain the milieu's enrichment (Brousseau, 1997) along the proof processes. Durand-Guerrier (2008) also stresses the importance of the manipulation of objects in order to make mathematical practice fertile. This viewpoint seems to converge with Weber & Alcock's position:

> "Just as most streets in a town intersect many other streets, at any given point in a proof, there are many valid inferences that can be drawn that might seem useful to an untrained eye […]. Hence, writing a proof by syntactic means alone can be a formidable task. However, when writing a proof semantically, one can use instantiations of relevant objects to guide the formal inferences that one draws, just as one could use a map to suggest the directions that they should prescribe." (Weber & Alcock, 2004, p. 232)

Obviously, every argumentation does not lead to proof, since the rules of the game are different in these two activities. In particular, in geometry, it is likely that the important gap between the different semiotic registers makes it more difficult to shift from an argumentation game to a proof game. As stated by Balacheff (2008, p. 509), it is necessary to bear this semiotic thinking in mind in order to understand Duval's approach. However, as we pointed it out in our examples, the assumption of an impassable gap between proof and argumentation is likely to hinder students' validation attempts. In particular, when validation is not immediate (we mean that it does not directly derive from the manipulation of the definitions of concepts involved in the statement of the proposition to be proven), it is often necessary to work on the content of the propositions. From a mathematical activity viewpoint, proof production seems to go with the familiarisation with mathematical objects.

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EXPERIMENTAL MATHEMATICS AND THE TEACHING AND LEARNING OF PROOF[1]

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Aim of this paper is to discuss the role of experiments in mathematics for the teaching and learning of proof. I summarize some research findings from basic research studies and from teaching experiments. The examples comes from teaching experiments at all school levels on space and geometry by means of classical resources although some of the findings might be expanded to other subject areas and to ICT. They allow to frame the topic within the international literature on conjecture production and proof construction: they support the advantages of experimental approaches to the teaching and learning of proof and, at the same time, point at some critical points to be controlled in order to design appropriate teaching interventions.

INTRODUCTION

A growing interest is shown, at the international level, for the development of approaches to mathematics where the active participation of students is encouraged within a laboratory setting, with hands-on activities. The emphasis on experiments, manipulation and perception, measurement and examples is shared by the approaches developed within ICT environments (both DGE and CAS) and within classical technologies (straightedge, compass and ancient instruments). This experimental approach, where exploration plays a major role, seems appealing for students, who quite often find the evidence offered by a particular experiment much more convincing than a rigorous proof (Jahnke, 2007) and are bored by the request to produce also mathematical arguments. Hence, the appeal of experimental approach might be suspected of obstructing the development of mathematical styles of reasoning: some believe that hands-on activities are useful in either science centres or mathematical festivals, where popularization of mathematics is in the foreground, whilst are not useful and may be even risky in the mathematics classrooms, where the construction of mathematical meanings is at stake. In other words, many mathematics teachers are afraid that the need of mathematical proofs and of deductive arguments is put in a difficult position if experiments are given too much space in the mathematics classroom, at least in secondary schools. In the following, after a short review of literature, I present some effective experiments at all school levels where experiments and exploration have been combined with theoretical aims like conjecture production and proof construction.

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SOME STUDIES CONCERNING PROVING IN THE MATHEMATICS CLASSROOM.

The literature on proof and proving is large and encompass different aspects. In the recent book on "Theorems in School" edited by P. Boero (2007), the following aspects are highlighted: the historical and epistemological dimension; curricular choices, historical traditions and learning of proof (including two national case studies); the cognitive dimension of the relationships between argumentation and proof; the didactical dimension including both teacher education and classroom practices. In the chapter authored by Bartolini Bussi et al. (2007), a mathematical theorem – for didactical purposes - is conceived as a system of statement, proof and theory. All these three components are important: the theory as a system of shared principles (sometimes called postulates or axioms and definitions); the statement as the result of a conjecturing process, where exploration through experimental activity is in the foreground, the proof as a sophisticated argumentation that is, on the one hand, connected with the conjecturing process, and, on the other hand, consistent with the reasoning styles of mathematicians (e. g. deduction from the accepted principles). This approach is consistent with Jahnke (2007), who speaks about 'local theories', i. e. small networks of theorems based on empirical evidence and claims: "There is no easy definition of the very term ''proof'' since this concept is dependent of the concept of a theory. If one speaks about proof one has to speak about theories, and most teachers are reluctant to speak with seventh graders about what a theory is". And Arzarello (2007) adds: "A statement B can be a theorem only relative to some theory; it is senseless to say that it is a theorem in itself: even a proposition like "2+2=4" is a theorem in a theory A (e. g. some fragments of arithmetic)".

In the above sense, it is possible to speak about theorems also within primary school, provided that the theories are "germ theories", drawing on empirical evidence, with the expansive potential to capture more and more principles. Germ theories, with principles constructed on empirical evidence, are crucial up to 8th grade; later, accordingly to curriculum, the reference to more and more structured mathematical theories is possible. So, for instance, in the teaching experiments below, the reference theory from grade 11th on is expected to be elementary geometry (either 2D or 3D) with some additional parts concerning either isometries or conic sections.

The links between argumentation and proof from a cognitive perspective have been carefully analysed by Pedemonte (2007) who devoted her doctoral thesis to the development of the idea of cognitive unity, meant as a kind of continuity between the production of a conjecture and the construction of the proof. Experimental research shows that proof is more 'accessible' to students if an argumentation activity is developed for the production of a conjecture: in fact this argumentation can be used by the student in the construction of proof by organising in a logical chain some of the previously produced arguments. These studies may have important consequences on the teaching and learning of proof: to explain why rote learning of ready made proofs is not successful for most students; to select suitable problems, which might foster conjecture production before proof construction; to understand why in some cases proving remains difficult in spite of the previous conjecturing process.

In the following sections I shall quote very quickly some experiments where conjecturing and proving were promoted, at different school levels and with different organization.

EXAMPLES FROM LONG TERM TEACHING EXPERIMENTS

In the attached table, some paradigmatic examples are quoted from long term teaching experiments developed as coordinated studies by different research teams. All the tasks concern a conjecture production before proving construction. They appear, however, different from each other.

Three tasks (tasks 1,2,3) concern individual activity, to be solved in paper and pencil setting; three tasks (tasks 4,5,6) concern small group activity, to be solved in writing after the exploration of a material object. The exploration is free in the case of sunshadows (task 4), whilst it is guided by sheets or by the teacher himself in the two cases from secondary school (tasks 5 and 6). The tasks 1 and 3 are construction problems: they require to produce a drawing and to justify the validity of the used method. The expressions "Explain ....." mean, in a language accessible for young learners, to justify the drawing process with reference to a shared (germ) theory. The task 2, on the contrary, seems to be given in a discursive way. Yet the explanation requirement with reference to a shared (germ) theory is implicit, as a part of the tacit rules shared within the classroom involved in these experiments. In the last three tasks proof is not explicitly required. Actually the focus is on the production of the conjecture. This is an intentional choice, because the problems are quite demanding. The tasks 4 and 6 concerns 3D geometry, that is usually not well mastered by secondary school students. The task 5 is difficult: the conjecture concerns a rotation around the lower point (O) in the Fig. 3. Actually to recognize it, it is necessary to "see" two line segments (OP and OP') that do not exist, to realize that they are always equal and, more generally, to be able to "see" invariants during the motion. The teachers, for the tasks 4, 5 and 6 had designed, according to the shared theoretical framework, an intermediate step where to collect and discuss the conjectures, before entering the proving process. In the task 4, students are explicitly requested to produce a general statement. This expression was used in those classrooms to foster the production of statements with universal quantifiers (all, always, and so on) and hopefully in conditional form (if ... then) to pave the way towards the construction of a proof with specified hypothesis and thesis.

Gr.	GERM THEORIES	CONJECTURES - PROBLEMS	SETTING

	(REF)	*THE TASK - TO BE SOLVED IN WRITING*	MATERIAL
1. Gr.2 - 8	The invariance of alignment in perspective drawing (Bartolini Bussi, 1996)	The centre of a table drawn in central perspective. *Draw the small ball in the centre of the table. You can use instruments. Explain your reasoning.*	Individual task (Fig. 1)
2. Gr.2 - 8	Motions of geared wheels (Bartolini Bussi et al., 1999)	The motion of trains of toothed wheels. *What about three wheels geared with each other?*	Individual task No material
3. Gr.4 - 8	The equality of the distance of the centres of two tangent circles to the sum of radii (Bartolini Bussi et al., 2007)	The drawing of a circle tangent to two given circles. *Draw a circle with a radius of 4 cm tangent to the given circles (radii 3 and 2). Explain carefully the method. Explain carefully why it works.*	Individual task (Fig. 2)
4. Gr.6 - 8	Mathematical model of sunshadows. Basic properties of lines, planes, parallelism and perpendicularity (3D geometry) (Boero et al., 2007)	The parallelism of sunshadows of sticks. *In recent years we observed that the shadowsof two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and of an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement.*	Small group work. Pens, pencils, notebooks, rulers, to reify lines and planes
5. Gr.11	Elementary geometry (3D geometry). Definitions and properties of isometries. (Bartolini Bussi & Pergola, 1996)	The isometry (rotation) produced, as a correspondence, by a pantograph. After a guided exploration of the pantograph. *If P and P' are two writing points, draw two corresponding figures. Which are the common properties of the two figure? Can they be superimposed? Does it exist a simple motion which superimposes them? Describe it.*	Small group work. A pantograph with graphite leads in P and P' (Fig. 3).
6. Gr.12	Elementary geometry (3D geometry). Metric definition of conics. Equations of conics (Bartolini Bussi, 2005)	The conic obtained by cutting a cone in a suitable way.The task is given orally by the teacher. *You have to obtain an important property of parabola [...]. As you see,* [the parabola] *is in a 3D space, on the surface of the cone [...]. you have to discover the relationship between the green line segment* [AE in the Fig. 4] *and this line segment* [EB in the Fig. 4].	Small group work. A 3D model of a cone with a normal cutting plane (Fig. 4).

Table 1. Some paradigmatic examples.

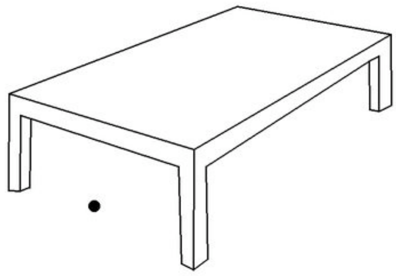

Figure 1. The small ball and the table

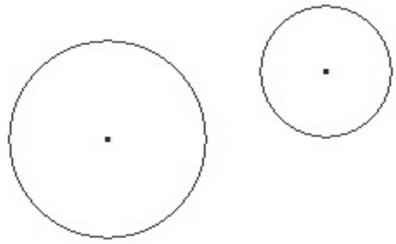

Figure 2. The two circles and the tangent circle

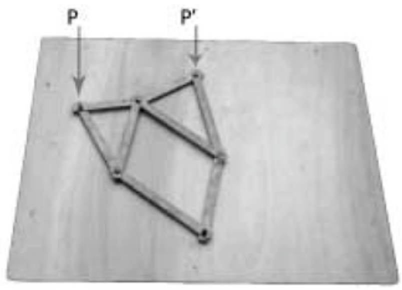

Figure 3. The pantograph

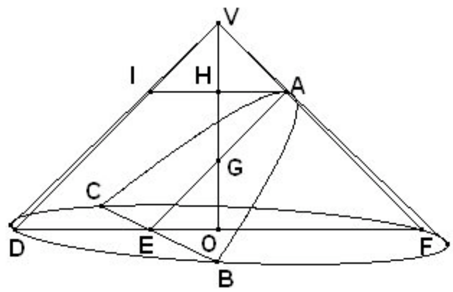

Figure 4. The parabola

At all ages, the dynamic exploration of a suitable problem situation has a crucial role both at the stage of conjecture production and during the proof construction. In particular, as to the conjecture production "the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, which will be subsequently detached from it and then "crystal" from a logic point of view ('if .... then')"; and as to the proof construction, "for a statement expressing a sufficient condition ('if ... then'), proof can be the product of the dynamic exploration of the particular situation identified by the hypothesis" (Boero et al, 2007, p. 249 ff.). This phenomenon has been observed by Boero et al. (2007) for the task 4 about sunshadows, by Bartolini Bussi & Pergola for the task 5 about the pantograph (Bartolini Bussi & Pergola, 1996) and in other ongoing experiments on either transformation or curve drawing devices. As concrete manipulation of materials is not spontaneous and guaranteed with elder students, who had already spent years to learn (or better to be taught) that mathematics is just a mental activity, the teacher has to foster it in a very coercive way: concrete exploration in demanding tasks is quite often the only effective way to promote dynamic exploration. Younger pupils, on the contrary, were accustomed to explore and to evoke exploration when no concrete object was available.

THE PROCESSES

The six situations above, although in different modes, have been designed to foster cognitive unity between the conjecturing and the proving phases. I shall not try to summarize here the observed processes concerning them all: they are complex, long standing, different (also for students' age) and all available in the international literature. Rather I shall illustrate another simple case of conjecture production and proof construction at secondary school level (from grade 10 on, according on the curriculum), concerning a curve drawing device. I shall narrate the stories of dynamic exploration that show up when secondary school students are given this curve drawing device to foster reasoning, conjecturing and proving (another example is discussed by Bartolini Bussi, in press).

I shall collect some evidences from the field notes of the exploration sessions in both school classrooms and the Laboratory of mathematical Machines (www.mmlab.unimore.it), to highlight the patterns that emerge. The two parts of the fig. 5 show (on the left) a drawing from the XVII century treatise by van Schooten (1657, p. 339) and (on the right) a photo of the brass copy reconstructed on a wood platform (40 cm x 40 cm) by the team of the Laboratory of Mathematical Machines at the Department of Mathematics of Modena, to be used with secondary and university students. The students are supposed to know some early properties of conics, e.g. the string and pencil drawing of an ellipse (together with the ellipse metric definition).

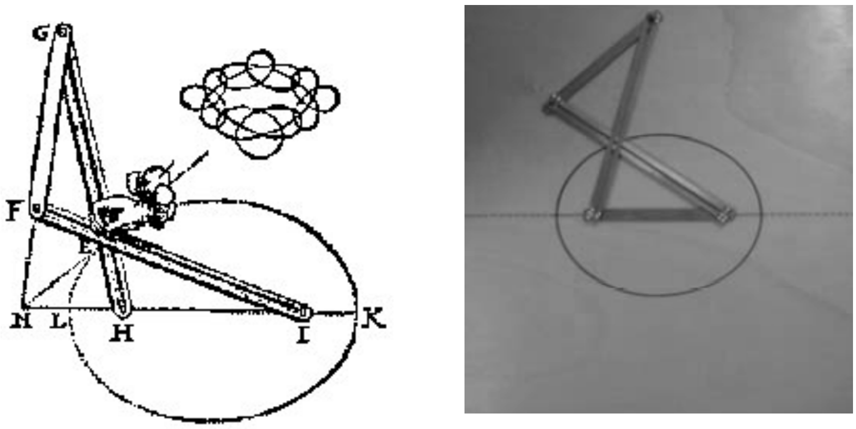

Fig. 5a and 5b: Van Schooten's Ellipsograph

There are several ways to explore the artefact (in order to produce a conjecture and to construct a proof of the conjecture) that span from strongly to weakly guided ones. In general, strongly guided exploration is suitable to the short term sessions (at most 2 hours, including the introduction and the conclusion of the hands on activity, Maschietto & Martignone, in press) which take place when a classroom come to the Laboratory, whilst weakly guided exploration is suitable to classroom activity, when the teacher plans to spend more time on the same topic. Actually with a weak guide, the time may expand, not matching the time constraints of a short visit to the Laboratory.

A) Strongly guided exploration. Students are given a worksheet where a layout of the artefact is drawn with coding letters (examples: http://www.mmlab.unimore.it/on-line/Home/VisitealLaboratorio/Materiale/articolo10005163.html) and are suggested to identify the fixed points, the trajectories of the moving ponts (e.g. G and F), the length of the bars, and so on. After this exploration, they are asked to conjecture the name (if any) of the trajectory of the point E (intersection of GH and FI in the fig. 5a) tracing it with a graphite lead on the wooden platform. The drawing is soon recognized as an arch of an ellipse and the conjecture is produced. Then the process of proof construction is to be started. We shall comment it later.

B) Weakly guided exploration: students are given the artefact and the information that it may draw curves; they are given the burden to produce conjectures and to prove them. A graphite lead to trace the trajectory of points is available with no special emphasis on this experiment: they can decide to use it or not. The artefact is without coding letters (Fig. 5b) and actually the need of coding may be one of the outcomes of the exploration to understand each other (Bartolini Bussi & pergola, 1996). When the students explore for some minutes the motion without drawing the arch, they may recognize a well known (although hidden) figure. HIGF (fig. 5a) is an isosceles trapezium with diagonals (HG and FI) and sides (FG and HI) given by brass bars, whilst the bases FH and GI have a variable length and are not reified by bars. The figure is not trivial to be noticed, as the two bases are not visible. Usually the students rotate G around H and observe the figure. Sometimes they seem fascinated by this rotation. They stay silent for minutes. They try to look at the artefact from different perspectives, also standing and miving around the table. They assume strange postures, twist their necks to follow the motion, point at the bars and follow the motion with the finger in the air, move the bars forward and backward to look for invariants and test them stopping the continuos process. In the small group work, sometime a conflict arises, when the speed of the motion controlled by the actor does not match the exploration planned by the observer. At one point they "see" the trapezium and notice that EG = EI and FE = FH. When a student has "seen" the trapezium, this figure is immediately shared with others. When the trajectory of E is eventually drawn they have at disposal what they need to link the conjecture with the metric property of ellipse.

I have described two 'antipodal' exploration processes with a lot of mixed cases in between. The weakly guided one is enjoyed by experts. The strongly guided one suits novices' needs to avoid frustration: it aims at encouraging to handle the artefact and at scaffolding the process. In both cases the demanding part is not the conjecture production, especially when drawing by the graphite lead is encouraged. Actually, as soon as the user draws the curve, the conjecture springs up, because only a limited set of curves is known by students: it is neither a circle nor a parabola nor an hyperbola, hence it must be an ellipse. The demanding task in this case concerns proof construction. This situation is different from the one of the tasks 4 and 5 above, where also conjecturing is really demanding.

In the strongly guided exploration, the worksheet suggests some ways to explore the properties of the artefact. Yet, in order to notice the properties, measuring by rulers is suggested. Measuring requires to stop the motion and to transform the experience of continuous motion into the observation of a finite set of frames. The focus risks to be on measuring parts of still figures.

In the weakly guided exploration, the focus shifts on the observation of dynamically changing shapes and their invariants. The students have to move and observe. Their process seems time wasting and not effective and has to be monitored by a walking teacher who moves from one group to another showing how to explore the artefact, with changing speeds and, maybe, no word. The initial 'weak' guide seems to require a stronger teacher's control. The students do not need (and usually do not wish) to measure bars by a ruler. As soon as they notice some invariants, they use their hands: they pretend to pick up the line segment EG between forefinger and thumb and to rotate it until it matches EI. They repeat the action on the pair FE and FH. Silent gestures seem to be enough to convince them. Maybe words and deductive chains are missing. Writing and justifying (by symmetry, for instance) the equality:

$$HE + EI = HE + EG = HG$$

that represents the metric property of ellipse with foci H and I is the boring counterpart of a relationships discovered by making "infinitely many" experiments, during the continuous motion of G around H.

In both cases of exploration, if the drawing is produced too early, the attention is focused on the final result of drawing rather than on the dynamical process of drawing. I shall consider this later.

There is a difference between the strongly guided exploration, that foster the production of statements concerning pointwise construction of the trajectory and the weakly guided exploration, that foster the production of statements concerning the global construction of the trajectory by a continuous motion. This difference is epistemological and mirrors the ancient pointwise construction of curves and the modern (as from the 17th century) construction of curves by a continuous motion of a machine. In the pointwise construction, there is a gap between the statements concerning a particular point E obtained when the artefact is in a given position and the generalization to a whichever point of the trajectory. This gap might obstruct the proof construction, requiring additional arguments.

The situation is different, yet recalls the one analysed by Pedemonte (2007) and concerning the construction of proofs by mathematical induction. She analysed the sum of the interior angles of an n-sided convex polygon, but the reasoning might be applied to many cases of induction. The well known formula: (n - 2) times 180°, may be conjectured in at least two ways, that draws on experimental activity and that are called: result pattern generalization (the cases of n-sided convex polygons are analysed separately, adding the measures of the interior angles, for n=3, n=4, n=5 and so on); process pattern generalization (from an (n-1)-sided convex polygon, for n=3, 4, 5 and so on, a new n-sided convex polygon is obtained by the juxtaposition of a triangle, whose sum is 180°).

The result pattern generalization does not help much to construct the proof by mathematical induction, because the argumentations used have no counterpart in the proof. On the contrary the process pattern generalisation paves the way towards the proof, showing how it is possible to shift from n-1 to n. Pedemonte (2007) says that in the second case there is a structural continuity between the conjecture production (by argumentation) and proof construction (by induction). Students may succeed in proving the conjecture also after a result pattern generalization, but they must reconstruct a suitable argumentation that links the conjecture to the proving process.

The shift to the analytic frame suggested in the Laboratory worksheets is an intentional break of the structural continuity, because the analytic frame is supposed to be the familiar context where conics are studied in secondary schools.

DISCUSSION

Some conclusions may be drawn from the quoted examples and research outcomes. First, there are good reasons to believe that conjecturing through exploration before proving might be very useful. Yet, when conjecture production is too fast, it might offer no element to be used in the proving process. Hence it is useful to look for strategies that slow down the conjecture production and encourage effective exploration of the problem. The time spent in conjecture production is not wasted and may be recovered in the proof construction. Second, it is not possible to give general rules about which exploration is effective in the conjecture production. In the last example, I have contrasted strongly guided and weakly guided explorations, which are only two examples of a very rich set of possibilities. What to choose in a classroom situation? The teacher's decision has to be contextualized and depends on a lot of issues: the time constraints, the curriculum, the students' qualifications and so on. This last issue is related to teacher education. The teacher's knowledge in order to design and to manage in the mathematics classroom this kind of activities is complex and does not fit in the space of this paper. A systemic approach to teacher education is now in the foreground in the literature on didactics of mathematics.

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THE ROLE OF CONTENTWISE ARGUMENTATION FOR THE CONSTRUCTION OF MATHEMATICAL KNOWLEDGE

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This pre-study for my PhD-project arises from my examination paper dealing with the comparison of two models for the analysis of epistemic processes: the RBC- and the SVSt-model. The analysed scene is about two students developing strategies to compare infinite sets. Certain moments in their process of argumentation seem to be substantial for the construction of their new mathematical knowledge. But without a tool for analysing the depth and the concentration of their argumentation, this connection cannot be demonstrated. In a first analysis, the Toulmin`s scheme for analysing argumentation processes turns out to be an appropriate tool to characterize the depth of argumentation and different types of concentration of arguments.

STATE OF RESEARCH AND RESEARCH QUESTIONS

Most of the didactical studies about argumentation are connected with the discussion about proofs in mathematics education. The activity of proving as `high art´ of mathematics is confronted with a low need for proofs and a lack of understanding on the part of the students (Knipping 2003, Reiss 2002). The discrepancy between the acceptance of learning and aims of mathematics education led to a discussion about changes in the view of proving in mathematics education (Dreyfus 2002). Krummheuer instead proposes to focus more on arguing than on proving as arguing is important as

- didactical category for conceiving of proving or activities similar to proving
- as pre-form of proving specific to school and/or
- as `modern view´ of the understanding of proving in mathematics.

(Krummheuer 2002, 247)

In his and other newer publications displaces the concept of arguing the concept of proving more and more. Especially, Krummheuer & Brandt (2001) assume a constitutive function of arguing for learning mathematics. Without participating in collective argumentation learning mathematics in school`s interaction would not be possible. Therewith, the epistemological view is picked up (Jahnke 1978, Knipping 2003). Proving serves not only for assuring knowledge but for develop new mathematical knowledge and understanding. Although the knowledge developing function is unquestioned, the assignment with theories of construction of mathematical knowledge is missing in the didactical research. In the following, a first attempt therefore is proposed.

The RBC-model developed by Dreyfus et al. (2001) was used several times during the last years for analysing processes of construction of mathematical knowledge. In her studies about interest-dense situations, Bikner-Ahsbahs (2003) designed the SVSt-model to describe epistemic processes. In my examination paper (Cramer 2008), both models were used to analyse the same scene. Thereby, the missing of a contentwise level in both models turned out to be a deficit. The quality of constitutive moments of the epistemic processes could not be identified until viewing the content component of the argumentation. From that view, the analysed scene can be described as a continuing process of argumentation of different depth and concentration. The Toulmin`s scheme is an adequate tool for analysing the depth of argumentation. In a re-analysis of the scene, it has to be verified if the concentration of arguments can be characterized with the Toulmin`s scheme. The central questions this article is about are: How can the Toulmin`s scheme be used to characterize the concentration of an argumentation? Which conclusions can be drawn by this tool? Which hypotheses about the correlation between processes of argumentation and epistemic processes can be proposed?

THEORETICAL BACKGROUND

How to describe epistemic processes?

Key activities in constructions of mathematical knowledge are mental activities that cannot be observed directly. The RBC- and the SVSt-model try to solve this problem by using the term of epistemic actions. Epistemic actions are those actions that constitute the process of construction of mathematical knowledge and are observable by verbal comments or non-verbal actions.

The RBC-model

Construction of mathematical knowledge passes three stages: 1. the need for a new mathematical construct, 2. the development and 3. the consolidation of a new construct. Dreyfus et al. (2001) characterized three epistemic actions: recognizing, building-with and constructing. Constructing is the central step in processes of abstraction. To achieve a new mathematical construct, available elements of knowledge that are recognized as valuable were accumulated and integrated. Recognizing means realizing that a mathematical construct already known is inherent in a mathematical situation, important for it or relevant in any other sense. Building-with is expressed by combining of known elements to achieve a certain aim. The classification of these actions is strongly individual. The same action can be building-with for one student and constructing for another depending on whether the elements of a construction have been available before or not. The epistemic actions can be nested in another or proceeding parallel. There are some mechanisms very effective for consolidating new mathematical constructs: reflecting about a new construct, using the new construct in phases of building-with and realizing the relevance of the construct for another construction. The two last mechanisms show that not only the epistemic actions in one construction can be nested but also the constructions of different mathematical constructs as well. As consolidation of a construct is part of construction of mathematical knowledge, the model is sometimes called the RBC+C-model.

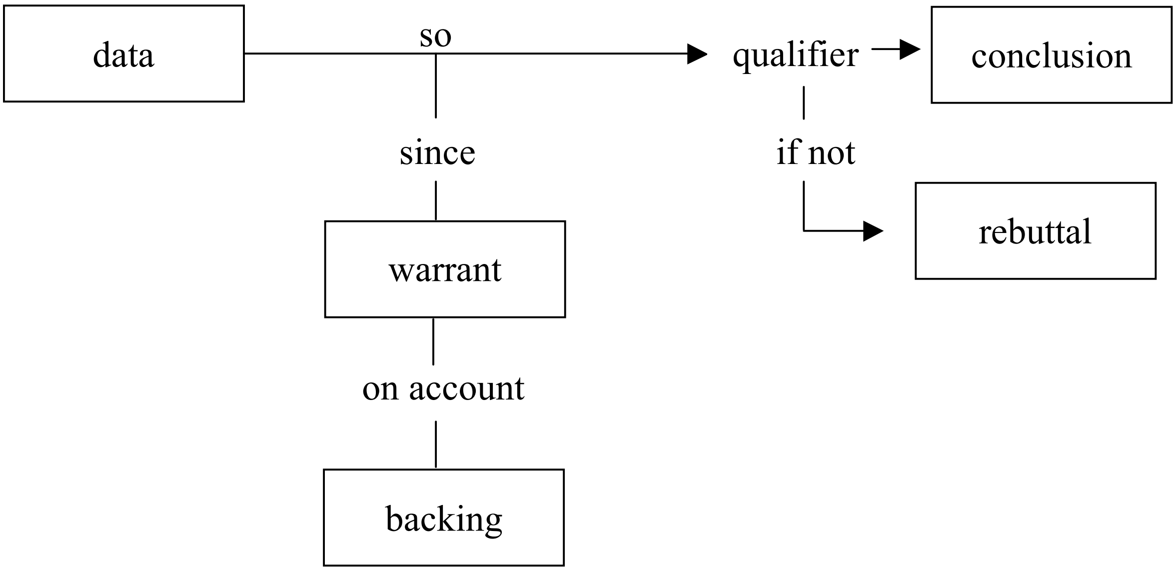
The SVSt-model

This model (Bikner-Ahsbahs 2005) differentiates the epistemic actions collecting, combining and seeing structures. Collecting of senses provides a basis for constructing knowledge. Examples, counter-examples, connotations, patterns, prototypes, solutions, approaches, facts and so on can be collected and represents the material for the following construction. Senses are combined by seeing connections between those senses, by reasoning or by summarizing the results. Seeing structures is characterized by discovering mathematical structured, patterns, laws or prototypes of solving methods. Seeing structure does not only mean the realization of a structure but also concretising, reasoning and testing a structure, re-structuring of elements of an example or a sector.

Three ideal types of epistemic processes in interest-dense situations in classrooms could be reconstructed. The cascaded type is characterized by separated phases of collecting and combining that lead consecutively into a phase of seeing structure. The phases are initiated by the teacher. The expectations about the student`s activities are clarified on the one hand, but on the other hand there exist open options for the students. The spiralled type starts with vague situations containing a problem. The typical pattern of these situations consists of discovering structures and concretizing and/or testing this structure by old or new examples. The enlargement of the amount of examples is part of this ideal type and heightens the chance to realize more aspects of a structure. Phases of collecting and combining were passed several times. The importance of combining increases constantly and constitutes the basis for seeing structure that appears spontaneously and discontinuously. The students form the epistemic process more and more independently. The third ideal type is relevant for situations that amass solutions prepared at home.

The Toulmin`s scheme

To analyse processes of argumentation Toulmin (1958) developed a scheme that differentiate elements of an argumentation regarding their function. The conclusion is the statement that has to be reasoned. Unquestioned facts the conclusions is led back are named as data. Data and conclusion are part of every argumentation. If the step from data to conclusion has to be explained intensely, warrants have to be declared. To convince someone of the practicability of a warrant, it can be backed by more statements. To constrict validity of a conclusion, it can be specified by qualifiers or rebuttals.



Reasoned conclusion in an argumentation with more steps can turn into new data. If backings, warrants or data are questioned, these elements have to be reasoned in a separate process of argumentation before they can be used in the primal argumentation. In those complex processes of argumentation, Krummheuer & Brandt (2001) call these separate argumentations lines of argumentation. If several lines of argumentation deal with the same conclusion, backing or contradicting themselves, the whole process is called a cycle of argumentation.

ANALYSIS OF DATA

Mathematical background: strategies to compare infinite sets

In a study, Dreyfus & Tsamir (2002) identified five different strategies of students to compare infinite sets:

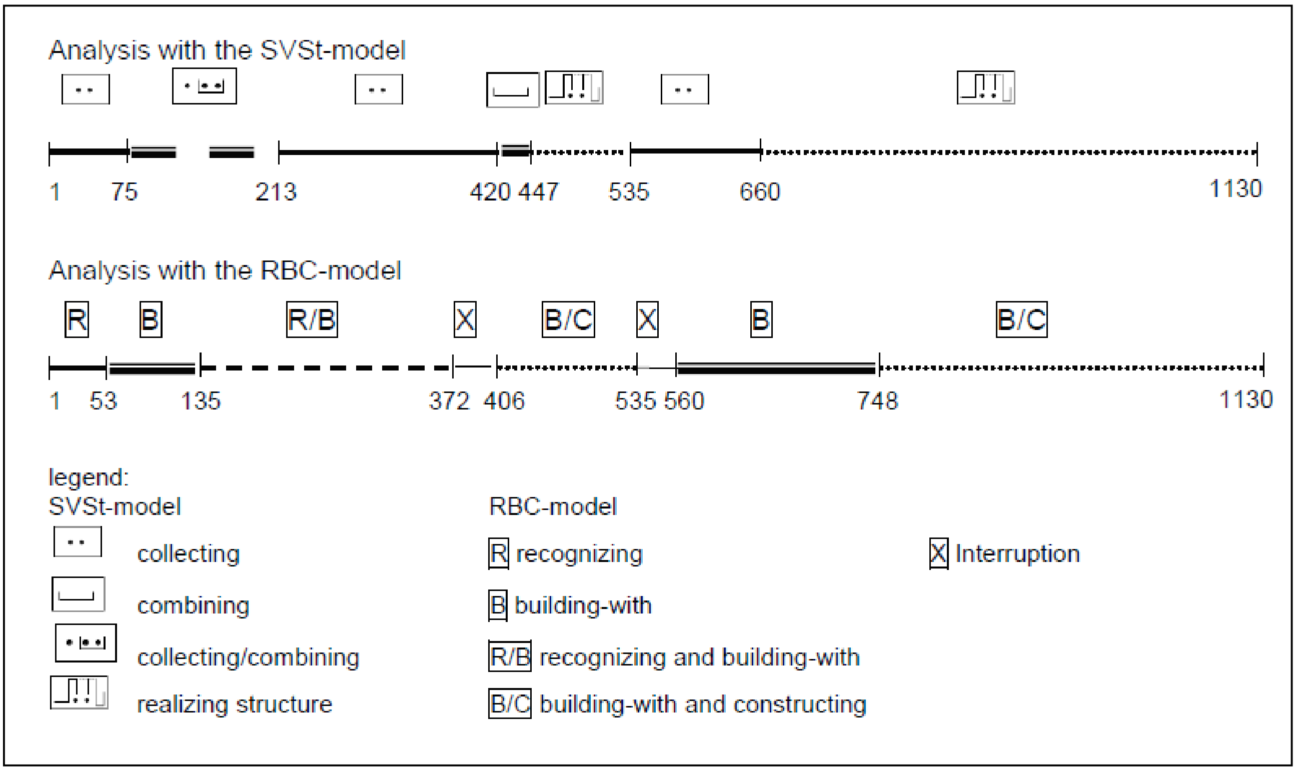
1. “single infinity”: Infinite is always infinite, therefore all infinite set are equal.
2. “intervals”: The one set is regarded as ordered sequence and tested for intervals. The sequence with more and larger intervals has fewer elements.
3. “inclusion”: The proper subset of a set has always fewer elements.
4. “incomparable”: Infinite is an undefined term. Therefore, infinite sets can not be compared.
5. “one-to-one-correspondence”: If a bijective function between the sets exists, they are equal.

The analysed situation

The analysed scene is extracted from a teaching interview with two high-capacity 15 year old students who examine the question how infinite sets can be compared. At this age, students do not have any experiences about infinite sets or transfinite numbers. Therefore, this problem is appropriate to explore processes of construction of new mathematical knowledge. The participation at the interview taking place out of school was optional. Decks of cards representing different sets of natural numbers were available and allowed to arrange the sets in different ways. All sets were compared with the set of natural numbers.

Methodological approach

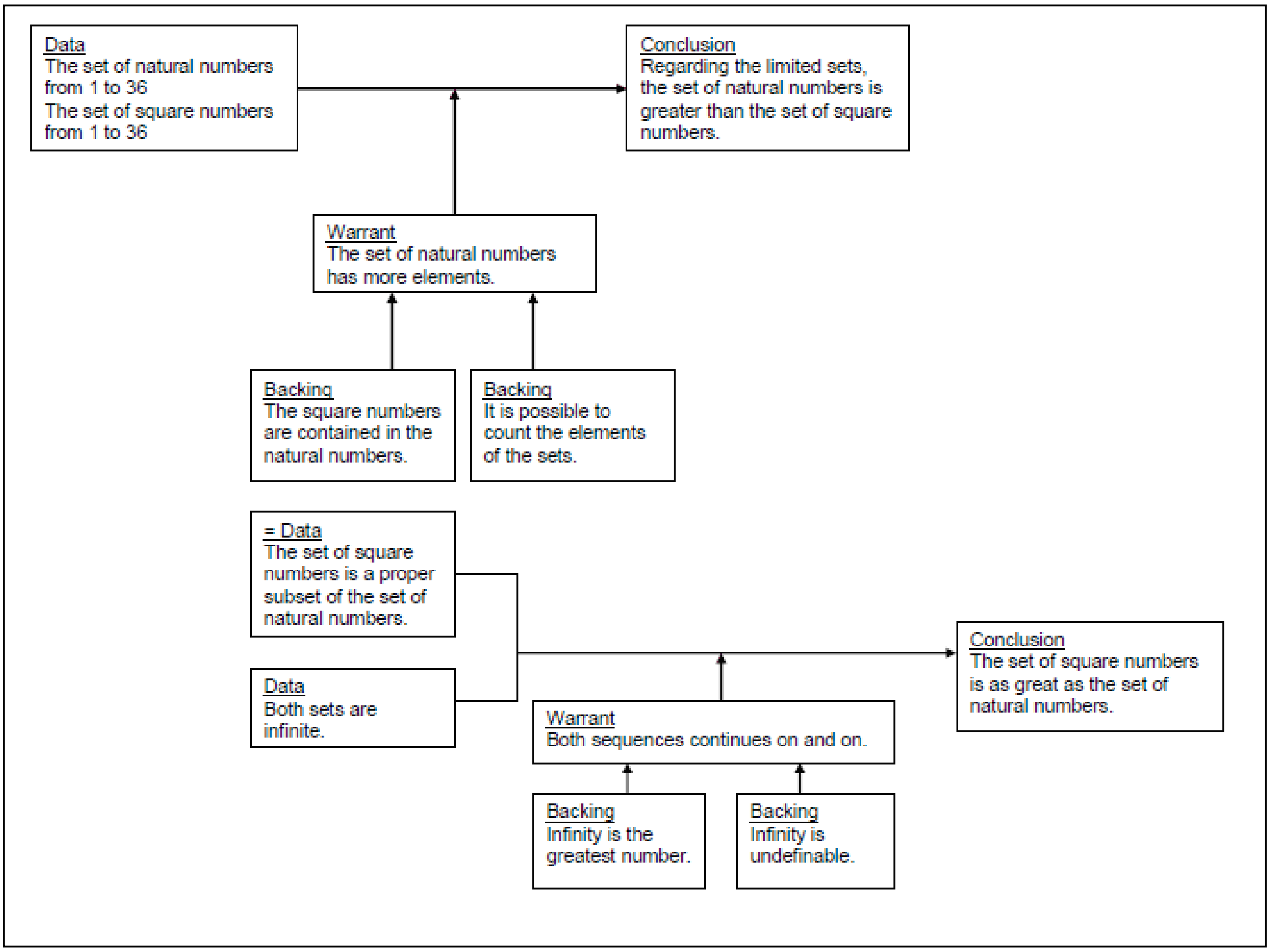
The present scene has already been analysed with the RBC- and the SVSt-model in my examination paper (Cramer 2008). In a first step, the different epistemic actions were represented by symbols in a diagram. These secondary data were compressed in a second step to characterize different phases of the interview by the hegemonic epistemic action. The SVSt-model contains the classification in phases characterized by collecting, combining, seeing structure as well as collecting and combining. The RBC-model assumes that the epistemic actions are nested. Although, different phases of hegemonic epistemic actions could be identified in my analysis. To compare both of the analysis, in the RBC-model were phases classified as well. In this way, the two diagrams are contrasted.



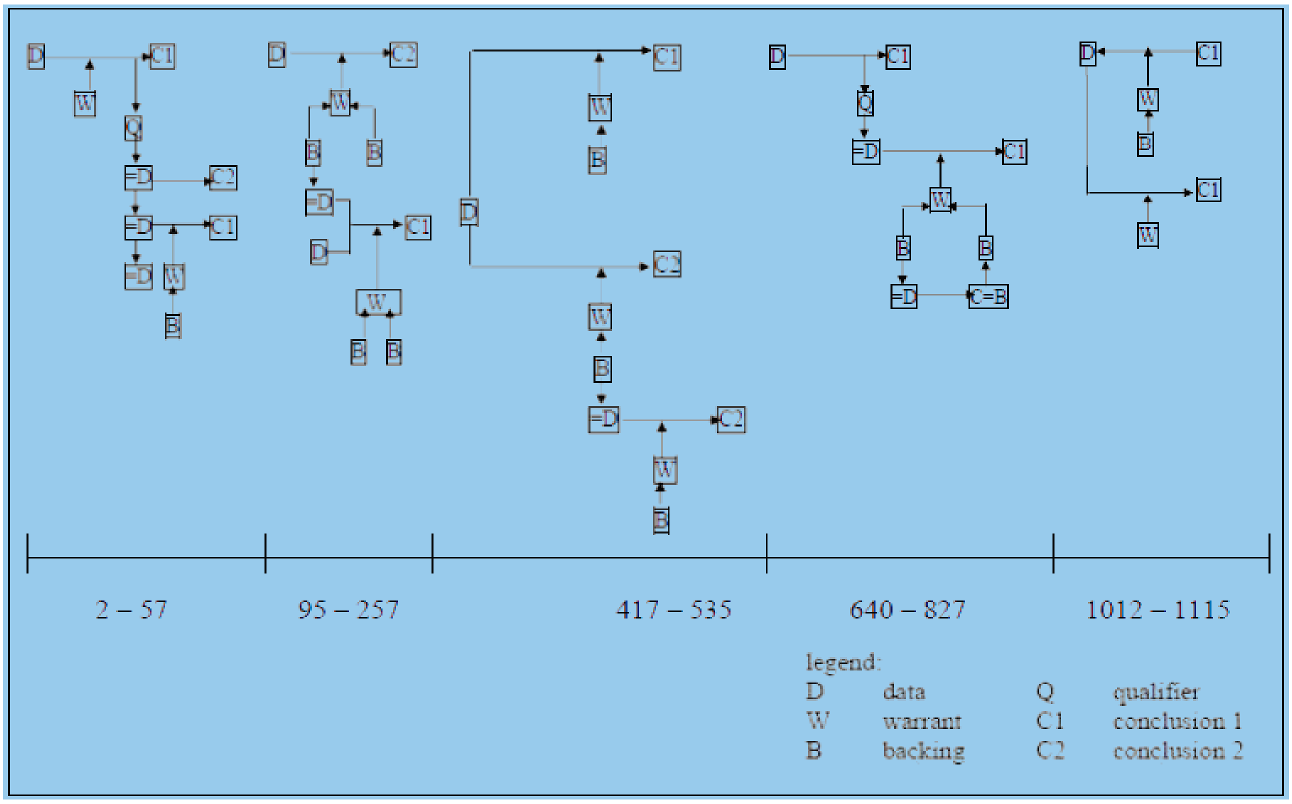
In a third step, the scene is now analyzed with the Toulmin`s scheme. These results are compared with the phases characterized by the hegemonic epistemic actions.

Analysis with the Toulmin`s scheme

The analysis with the Toulmin`s scheme reverts to the interpretation presented in my examination paper (Cramer 2008). To avoid being out of proportion, the analysis with the Toulmin`s scheme is represented in his macro structure showing the function of elements but not its content. The student's contributions have been recorded in a micro analysis. Exemplary, the second line or argumentation is shown in a micro analysis.



The macro structure of the argumentation process is presented by hiding the content of the elements and regarding nothing but the relations between the elements among another:



The process of argumentation starts and stays on a constant high level. Warrants, backings and even qualifiers were given from the beginning on. The process is a continuing cycle of argumentation with five main lines. Every one of these lines is already highly complex. All infinite sets that are explored in the interview are equal to the natural numbers. This right conclusion is named "conclusion 1". The wrong proposition that the explored sets would not be equal is named "conclusion 2". The identification of the two possible conclusions in the interview allows following the frequent changes between them.

The first four lines of argumentation emanates from data und develop conclusions on the basis of warrants and backings. The last line of argumentation is an abductive argumentation (Knipping 2003, 131ff.). From the conclusion on, the data is developed backwards. The step is reasoned by a warrant and backed by other statements.

In the first line of argumentation, a high concentration of data is conspicuous. In the second line, a high concentration of data as well as a high concentration of backings appears. The third line seems to be balanced. In the forth line, the argumentation structure becomes more complex. Thenceforward, the students always draw the conclusion, that the regarded sets are equipotent. They summarize several propositions that argue against (e.g. that the regarded sets are proper subsets of the set of natural numbers, that it is possible to indicate breaks in comparison to the set of natural numbers, …) and explicitly, they draw their conclusion in spite of these propositions. In the Toulmin-analysis, this is identified as an operator. To reason the warrant in the forth line of argumentation, another argumentation cycle emerges. On of the backings for this warrant is used as data. The drawn conclusions backs the other backing of the warrant.

Based on this description, two different types of concentration can be identified: In the first line, data are concentrated and in the forth line, patterns are concentrated. This description corresponds with the results of the SVSt- and the RBC-model. To explore a new mathematical problem, sufficient examples have to be available to deal with. In this case, there has to be a sufficient amount of infinite sets to test different conclusions. Testing happens by mentioned warrants and backings. Specifying and precising of an argumentation corresponds with phases of structure seeing in the SVSt-model and phases of building-with and constructing in the RBC-model.

The students use all strategies to compare infinite sets mentioned before during the whole interview. Furthermore, they use different ways to represent the number of elements of a set: Terms, graphs, diagrams and arranging in a certain order. Every time, the students use another way to represent the number of elements, they change their strategy to compare or their conclusion. This important role of the way of representing can be identified just by regarding the micro analysis of the scene. All those strategies and representations are used as warrants or backings.

CONCLUSIONS

The Toulmin`s scheme is developed to analyse the depth of an argumentation. This pre-study shows that a characterization of the concentration of an argumentation is possible as well, but not in the sense of counting the number of arguments during a certain time. The Toulmin`s scheme allows to describe the type of concentration by indicating the predominant elements of an argumenatation. Further studies have to clarify, if more types of concentration exist.

Furthermore, the Toulmin`s scheme in the micro structure is a helpful tool to maintain the content. The analyses of the epistemic processes describe the progress of the epistemic actions, but not their content. Though, key points in the construction of mathematical knowledge cannot be identified without the content.

A first comparison between the different analyses contains references to the connection between processes of argumentation and of epistemic processes. Specifying or precising an argumentation corresponds with certain epistemic actions in the SVSt- and the RBC-model. Hypotheses about typical types of processes of argumentation and construction of knowledge can be explored only by analyzing an amount of different situations. Thereby, information about assisting and obstructing conditions are potentially attainable.

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WHY DO WE NEED PROOF

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We explore teaching mathematicians' views on the benefits of studying proof in the basic university courses in Sweden. The data consists of ten mathematicians' written responses to our questions. We found a variety of ideas and views on the function of proof that we call transfer. All mathematicians in the study considered proofs valuable for students because they offer students new methods, important concepts and exercise in logical reasoning needed in problem solving. The study shows that some mathematicians consider proving and problem solving almost as the same kind of activities. We describe the function of transfer in mathematics, exemplify it with the data at a general level and present particular proofs illuminating transfer that were mentioned by the mathematicians in our study.

INTRODUCTION

The various functions of proof in mathematics and mathematics education have been discussed by researchers during many years and they have gained a wide consensus in the mathematics education research community (Bell, 1976; De Villiers, 1990; Hanna, 2000). Especially the functions of conviction and explanation have been in focus in the field (e.g. de Villiers, 1991; Hanna, 2000; Hersh, 1993). However, Weber (2002) states that besides proofs that convince or/and explain there are proofs that justify the use of definitions or an axiomatic structure and proofs that illustrate proving techniques useful in other proving situations. Lucast (2003) studied the relation between problem solving and proof and found support for the importance of proofs rather than theorems in mathematics and mathematics education for example from Rav's (1999) philosophical article. Lucast considers proof and methods for problem solving as in principal the same and states that proving is involved in the cognitive processes needed for problem solving.

According to the mathematicians in our earlier study, there are proofs that can introduce new techniques to attack other problems in mathematics or offer understanding for something different from the original context. For example, they mentioned the method of completing the square in deriving the formula for the solution of the second degree equation as useful in problem solving [1] (Hemmi, 2006). We decided to call this function of proof for *transfer* and we remarked that it had neither been in the focus in the research on proof in mathematics education nor involved in the earlier models about the functions of proof. It is close to and partly overlapping the aspect Weber (2002) describes but not exactly the same. Recently, Hanna and Barbeau (2008) have started to explore this function from a point of view of philosophy and mathematics education [2]. Also they stress that it has been overlooked in mathematics education research.

Extended information about various functions of proof communicates something about the meaning of proof in mathematical practice, and the consciousness of them should therefore be important for how newcomers experience the practice. Some students in our earlier study who had difficulties to follow and understand proofs that were presented in the lectures expressed for instance the lack of examples from mathematicians about connections between proofs and problem solving (see *condition of transparency, p. X*).

> Most often you don't have to be able to know anything of the proofs in order to solve problems. (Student – Intermediate course, 2004 in Hemmi, 2006)

They also advocated working manners and tasks where they could use the proofs in some ways in order to enhance their own engagement with proofs.

> I mean tasks in which you are supposed to calculate something using proofs. At least for me, it is easier to understand if I really use them for something. (Student – Intermediate course, 2004 in Hemmi, 2006)

Our recent study contributes to the field by exploring mathematicians' often tacit knowledge concerning the teaching and learning of proof in the practice of mathematics. In this paper, we first describe the function of *transfer* from the perspective of history of mathematics and then present an analysis of a pilot study with ten mathematicians concerning their views on proof, in particular with respect to the function of *transfer* in the basic courses [3] of mathematics in Sweden.

TRANSFER IN MATHEMATICS

Proof has not always been a natural part of mathematical activity. In the old cultures in Babylonia, Egypt and China, mathematicians seemed to be only interested in presenting results which could be used in different applications and not in the question of how these results were obtained. They might have done verifications of results also, but if so, they did not think it was worth while to write them down. With the Greeks, the deduction style of mathematics was born and the emphasis was put rather on the questions of truth, foundations, logic, and proving than on practical applications. Their work in geometry which we know from Euclid's Elements has since then been a model for scientific thinking. It was not until the 1900th century that proofs in algebra and analysis could be performed with the same kind of logical strength that was done in the Elements. Nowadays, proving has been almost a synonym for doing research in mathematics and an enormous amount of mathematical proofs are produced every year.

A natural question to ask is why the deductive style in mathematics has been so successful? Nobody can question the importance and usefulness of mathematics in the modern society, but do we need the proofs? It is only the very results in mathematics that are used in other sciences and, in the end, they are important for the production of all the facilities we see around us. We think the “market” should have forced mathematics to use the “handbook” style if this turned out to be as (or more) efficient as the ”deductive style”. For the Greeks it might have been possible to study proofs just because they thought it was an intellectual challenge, but in our society we think this is impossible.

However, the deductive style in mathematics has survived and been successful. One important reason for this is indeed that the proofs contain information of how to get other results and also often contain methods of calculation used for example in applications. As an example, consider Archimedes result about the volume of the sphere. It is of course interesting for applications to be able to compute the volume of a sphere, and with the formula in hand also some other problems maybe solved, e.g., the volume of a halfsphere. But without the proof it is hard to find formulas for the volume of other bodies. Archimedes described the method he used to find the formula, which may be seen as a form of integration and is interesting for other applications. It is a heuristic argument based on his law of the lever. The method contains a lot of information which may be used to reach far beyond the original problem. For other examples of theorems where the proofs are far more interesting than the results, see Rav (1999).

There is certainly a consensus among mathematicians that the proofs contain much more information than just the verification of the results, but how do they think about this function of proof in the teaching context?

METHODOLOGY AND THEORETICAL STANCES IN THE PILOT STUDY

In August 2008, we e-mailed to 16 mathematicians at various universities. We presented the aim of the study and invited them to share their thoughts with us concerning the following questions.

1. Why do you think that students in basic courses should become familiar with proofs and proving or do you think they do not need to do so and in this case why?
2. What specific proofs/derivations do you consider as central in basic courses which you have taught?
3. Are there specific proofs/derivations in the basic courses that teach students techniques, concepts, procedures, strategies or offer other tools that are useful in other contexts, for example in problem solving?
4. Are there proofs not filling the criteria in question 3 but which you in any way consider as central in the basic courses, in that case which proofs and why?

To encourage the mathematicians to response, we stressed that the answers would not need to be exhausting, it was enough to give some examples. Ten mathematicians from five different institutions e-mailed their answers. The responses varied both in length and in content. We had the possibility to contact the mathematicians and ask for complementary information.

We consider the mathematicians as *old-timers* in their *communities of practice of mathematics* (see Wenger, 1998). All the mathematicians in the pilot study had at least ten years experience of teaching and all of them have somehow been engaged in the teaching of elementary courses. Learning is conceived as increasing participation in the mathematical practice where proof is a central *artefact* with many functions (see Hemmi, 2006). According to the theory of Lave and Wenger (1991) artefacts and their significance to the practice can be more or less visible for the newcomers. This is called the *condition of transparency* of proof in the teaching of mathematics, i.e. how and how much to focus on various aspects and functions of proof and how and how much to use proof in doing and presenting mathematics without a focus on it as proof (see also Hemmi, 2008). This is one of the aspects in the conceptual frame that was created by combining the social practice approach with theories about proof obtained from the didactical studies in the field. The other aspects, relevant for this study, are the functions of proof of *conviction*, *explanation*, *communication*, *intellectual challenge*, *aesthetic* and *transfer*. We analysed the data with help of NVivo software by firstly relating the mathematicians' responses to the aspects in the conceptual frame. Then, we used an open approach and looked at the issues enlightening the function of *transfer* from various points of view and connected these issues to the themes described in the introduction (Weber, 2002; Lucast, 2003; Hemmi, 2006; Hanna & Barbeau, 2008). We interpret the mathematicians' utterances as representative of views belonging to the community, utterances that are influenced by the social, cultural and historical context of the same mathematics environment but also from other possible environments they are members of.

The aim of the pilot study is to investigate the diversity of ideas among mathematicians analysing a small sample in order to later explore a larger sample. This is why we cannot generalise the results and there is no use to give exact numbers of mathematicians talking about various themes. We make very little quantifications when reporting the results. First, we sum up the main reasons mentioned by the mathematicians for why they wanted to include proof in the basic courses. Then we provide some examples about utterances concerning the function of *transfer* at a general level. Finally, we present some specific proofs that according to the mathematicians involved this function.

RESULTS

All the ten mathematicians stated that students in the basic courses should become familiar with proofs and proving. This is interesting because in our earlier study which concerned only one department, most of the mathematicians said they did not deal with proof so much in the basic courses for various reasons (Hemmi, 2006). Yet, some of the mathematicians in the present study pointed out that there was no use to prove for example statements concerning limits of functions rigorously for the students studying engineering, chemistry or other sciences. One mathematician even stated that one should try to “serve up” mathematics for such students with so few proofs as possible and concentrate on applications.

The mathematicians gave various reasons for why proof is important to include in the curriculum for the basic courses. Some of them stated that proof helped to make visible the difference between school mathematics and university mathematics for the students and that inclusion of proof in the curriculum helped students to leave their preconceived interpretations about what mathematics actually is. Proof should be included in the basic courses because proof is the soul and the backbone of mathematics. It is the very idea of doing mathematics. According to one mathematician, working with some proofs also offered possibilities to discuss what proof is. This refers to the aspect of *transparency* (see Hemmi, 2008).

In line with our earlier study many mathematicians consider school mathematics as teaching students to apply rules they get through examples from the teacher or a textbook. According to the mathematicians, this manner does not lead to understanding of what mathematics is, “i.e., concepts and intuitive and logical reasoning about these concepts and their relationships”. Proof explains how the concepts are related to each other. This view refers to the function of *explanation*. Another reason the mathematicians gave was that proof connects all mathematics, without proof “everything will collapse”. You cannot proceed without a proof. This refers to the *verification* function of proof. Some mathematicians stressed that it was important to present proofs (or convincing arguments) for statements which are not conceived as evident by the students. This refers to the attempts to create possibilities for the students to experience the function of *conviction* of proof.

One mathematician stated that proof enhanced students’ interest towards mathematics by giving aha-experiences and also that students were curious about proof. The latter was confirmed by our study among university entrants. It showed that about 80 percents of students were interested in proof and wanted to learn more about proof when they came to the university (Hemmi, 2006). This refers to the function of *intellectual challenge.* One mathematician also pointed out that it was important to present some “beautiful proofs” even if he thought it was difficult to find such proofs suitable for the basic courses. This refers to the function of *aesthetic*.

Finally, one of the mathematicians talked about proof as useful in the learning of mathematical language. This refers to the function of *communication.*

All the functions mentioned above are interconnected and partly overlapping. Some of the reasons presented in this section that the mathematicians mentioned for why they wanted the students to meet proof in the basic courses are already connected to the function of *transfer*, the main target of this article.

Transfer at a general level

All mathematicians considered proofs more or less important in a manner that they taught students concepts and techniques needed in problem solving even if one of them mostly saw benefits at this level for other proving tasks. Some of the mathematicians stated that all essential proofs in the basic courses carried this function whereas others had difficulties to find examples of proofs involving this function at the basic level.

At a general level, many mathematicians mentioned that proofs helped students to learn *mathematical and logical reasoning* valuable in problem solving.

> If one becomes accustomed to study proofs one gets practiced with mathematical reasoning, something one can draw great advantages of in problem solving. Problem solving is an art of formulation. (M4)

> But they (the proofs) should also contribute to demonstrate and develop students' skills of logical reasoning. This is useful in many situations. One of the function of mathematics in the engineering program is this. (M8)

Yet, not all mathematicians considered this function of proof so important for engineering students as the one in the citation above.

Also the *understanding of generalisations*, especially with respect of the *models for problem solving* within mathematics or in applied sciences could be enhanced by studying proof according to some mathematicians.

> They have to start to argue for the solutions of the problems for example in applications that they present, show that they are correct, so they can work in a manner not just filling in numbers in given models but tackle new problems. (M10)

One mathematician talked about the value of proof for problem solving because they helped students to learn and *understand new mathematical concepts*.

> Mathematics is about defining concepts and to study how these concepts are connected. To understand the concepts you have to understand how they are connected to each other. […] From the proof one should learn something about the concepts involved in it. (M8)

Even *technical proofs* were considered as valuable by one mathematician as they helped students understanding of problem solving.

> Also the technical proofs are useful to do: the technique leads to better understanding of problem solving. (M1)

Here, the mathematician might mean that the proof techniques could be explicitly used in problem solving.

Proving and problem solving involved in each other

Some of the mathematicians stated already in their responses to the first question that they considered proving and calculating/problem solving as in principle the same activity (compare with Lucast, 2003). By highlighting this in the teaching they wanted to “demystify proof”.

> I don’t consider “poof” as something different from other mathematical activities – obviously it is about reasoning, calculating, being ingenious/creative, using one’s knowledge and experiences and then drawing conclusions. To prove the rule of squaring a binomial, to give an elementary example, is of course just to perform the calculation. (M9)

> I would like to extend the meaning of “proof” to refer to logical reasoning in general. In proofs one meets such reasoning in a concentrated form. But it is present also in problem solving and in mathematical discussions in general. (M4)

> There is no difference in principle between proving and calculation. When a student carries out a computation in several steps, then these steps is a proof of the statement that the final result is the answer to the question. It is important that students at all levels get the insight that it is always reasoning which is the core of mathematics. (M6)

However, these mathematicians considered working with specific proofs as valuable for problem solving.

Most of the mathematicians talked about *transfer* only at a general level but there were some examples of specific proofs that we found valuable to present in order to later explore their potentials for further studies.

Some examples of proofs that teach students concepts or techniques

The mathematicians mentioned a number of proofs and exercises as valuable for students in order to learn techniques applicable in *other proving tasks*. This refers to the function Weber (2002) writes about. We have gathered their suggestions in the following table.

The relation in Pascal’s triangle can be proved by **induction**
There are an infinite number of primes enlightens **proof by contradiction**
The square root of 2 is irrational. The students can then surely find other results where the number 2 is replaced by another integer.
n(n+1) is divisible by 2 , if n is a positive integer. The same proof techniques can be applied in other proving tasks concerning divisibility.
Is it true that the proposition P(x) holds for all real numbers x?” where P(x) is for instance an inequality. This trains the ability to see what is required of a proof, and that a refutation just needs a counter example which is very important in many proving tasks.

Open tasks. They encourage the willingness to investigate and make hypotheses – which then are to be proved or disproved.

The next citation is an example about how studying proofs or proving statements concerning the derivatives is seen to help students to become familiar with and learn to understand *new concepts and definitions*, in this case the notion of the limit of a difference quotient as a derivative.

> The derivative is defined as the limit of a difference quotient, and you get a geometric interpretation as the slope of the tangent, but you also have the technical interpretation as change of rapidity (in a broad sense). Next you derive (prove) the rules for the derivative of a sum, product, … and you derive the derivatives of the elementary functions. All these you may of course find in a table of formulas and you should moreover know them by heart, they are so important for the applications. But through studying the proofs you get opportunity to many times consider limits of a difference quotient, and in that manner consolidate the definition of the important notion of derivative. (M8)

The last quotation below is about the proof of the factor theorem. We find this proof as a good example of proofs at an elementary level that allow mathematicians to highlight importance of studying the methods and notions in proofs.

> We can begin with the factor theorem, that a is a root of the polynomial p if and only if p(a)=0. The theorem expresses for sure an equivalence and it is interesting to discuss that one implication is obvious while the other is deeper. If you look at the actual proof you then see that the proof gives a bit more than what the theorem states. Indeed, the proof gives us information about the rest even in the case where the rest is not zero. (M4)

As an example of a problem where the proof of the factor theorem could be useful, consider the following: Determine the remainder, without carrying out the division algorithm, when $x^4 + x^3 + x^2 + x + 1$ is divided by $x - 1$.

DISCUSSION

The study shows that the function of *transfer* is a natural way of thinking about proof for many mathematicians and all mathematicians express the importance of teaching proofs also in the beginning courses at university. Yet, one of them states that the students studying applied sciences do not need any proofs and some others that they do not need all the rigorous proofs. Only one mathematician did not think that proofs could be useful in problem solving at the basic level.

Some mathematicians wanted to look at proving and calculation/ problem solving in a similar way. The resemblance between proving and problem solving has been studied and discussed by Lucast (2003). This is an interesting point of view as we can also think the other way around, i.e., students can learn concepts and techniques in problem solving that they can use in proving tasks. We find it interesting to note that the connection between proving and problem solving is something fundamental in the area of *constructive mathematics*, where these two activities are considered to be not just similar but in fact the same (see Nordström & Löfwall, 2006). It could be fruitful to study the notions of proving and problem solving from the perspective of constructive mathematics in order to get more insight in their connections.

In school mathematics and also in the beginning courses at university it has been a tendency to avoid the word "proof" in order to not frighten the students (Hemmi, 2006). However, students lack discussions about what proof is and why it is needed. An important didactical question is how to in the best way highlight the connections between proving and problem solving in the teaching of mathematics (see *condition of transparency* p. x). Consider for example the following citation in p. x:

> To prove the rule of squaring a binomial, to give an elementary example, is of course just to perform the calculation. (M9)

The mathematician expresses here a view that proving, in this case, is just calculating but we could take it the other way around and discuss this calculation could be considered as proving.

We have shed light on the function of proof that we call transfer from historical point of view and explored mathematicians' pedagogical views on it. We have described transfer at a general level and exemplified some proofs where connections to problem solving can be made visible. It is clear that mathematical proofs are carriers of mathematical knowledge and there are various ways of enlightening this for students. However, we do not want to look at the function of transfer mechanically, even if there are situations where it is possible to just copy a proof technique to another proving task. We have to acknowledge that what experts consider as evident connections may be difficult to see for a learner. We have to study the personal constructions of similarity across proving and problem solving and also study this from learner's perspective (Lobato, 2003). However, our study shows that there is a lot to explore in university mathematics regarding the ideas from the mathematicians' personal experiences of proof in the learning and doing mathematics.

NOTES

1. Consider for example the following problem: Determine the centre and the radius of a circle $x^2+2x+y^2-4y=0$. It should be easier to solve it if one is familiar with the method of completing the square.

2. However, Hanna and Barbeau (2008) do not use the word *transfer* for this function.

3. With basic and elementary courses, we refer to the courses taught during the first semester, the *intermediate course* during the second semester and the *advanced courses* during the third and fourth semesters.

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UNDERSTANDINDG, VISUALIZABILITY AND MATHEMATICAL EXPLANATION

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In this paper I focus on Mathematical Explanation in Physics and I analyse its interplay with the concepts of understanding and visualizability. Starting from a recent contextual approach to scientific understanding (De Regt & Dieks, 2005) I will try to see how an historical analysis of the formulation of a particular theorem could help to clarify the role of understanding and visualizability in mathematical explanation. My test case will be Euler's theorem for the existence of an instantaneous axis of rotation in rigid body kinematics. In particular, I will argue that the specific concept of vector space, defining a new standard of intelligibility, offers a good perspective in order to underline the dynamical character of mathematical explanation and its essential role in mathematical education.

1. INTRODUCTION

Different authors agree that the problematic of explanation is deeply connected to the debate about the nature of understanding in science. At the moment the major accounts of scientific explanation such as the Unificationistic (Friedman, 1974; Kitcher, 1981, 1989), the Causal (Salmon, 1984), the Pragmatic (Van Fraassen 1980; Acrhinstein, 1983) do not offer a satisfactory definition of understanding within their theories. While the authors and the supporters of those theories affirm that their particular accounts of explanation provide understanding, the notion of understanding remains still vague and is the cause of a series of controversies between philosophers of science. It seems quite plausible that a good explanation in science *must* provide understanding. But what is *understanding*? Is it really this "aha!" experience we are confronted with after some cognitive experience? And how can a good explanation provide understanding?

In this paper I will focus on the very specific notion of mathematical explanation, and in particular on the notion of mathematical explanation in physics. As clearly expressed by Mancosu in his studies on mathematical explanation (Mancosu 2005, 2008), we can have two different senses mathematical explanation:

> In the first sense "mathematical explanation" refers to explanations in the natural or social sciences where various mathematical facts play an essential role in the explanation provided. The second sense is that of explanation within mathematics itself (Mancosu, 2008, p. 184).

Naturally, as pointed out by Shapiro (2000), mathematical explanation as intended in the first sense is connected to the more general problematic concerning the application of mathematics to reality and opens the mysterious problem of the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1967). However, leaving apart mysteries and ontological questions, many authors agree that it is possible to have a better comprehension of mathematical explanation of physical phenomena (MEPP) [1] starting from general discussions of scientific explanation and introducing an historical perspective (Tappenden, 2005; Kitcher, 1989). In this paper I will follow this line, getting my hands dirty *via* a bottom-up approach that starts from the mathematics itself. I will compare two different formulations of Euler's theorem for the existence of an instantaneous axis of rotation in rigid body kinematics and I will try to discuss the concepts of understanding and visualizability under the light of dynamical MEPP. I assume as a starting point that in both the formulations the mathematical machinery has an essential role: they represent two mathematical explanations of the same physical fact. Naturally, in such a contextual analysis, the arena of mathematical education is of primary importance and I will offer a perspective in order to work in this direction.

In a recent series of papers De Regt and Trout have discussed the notion of understanding in science (De Regt, 2001, 2004, 2005; Trout, 2002, 2004). My point will be that, contrary to Trout's idea that is impossible to give an objective epistemic role to understanding (Trout, 2002), some interesting ideas of De Regt's account could be utilized in order to study the role of visualizability and understanding in mathematical explanations. I hope this study will make clear that MEPPs have a dynamical character, and in some case the role of understanding in them could be studied if we have at disposition conceptual tools like visualizability. After all, a number of new studies and a sort of "renaissance in visualization" (Mancosu, 2005, p. 13) have emerged during the last years in philosophy of mathematics and cognitive sciences. The impetus in this sense has been given for the most part by the rise of visualization techniques in computer science, from which has clearly emerged the heuristic and pedagogical value of visual thinking [2]. Naturally, I stress again, my analysis implicitly focus on the importance of mathematical activity and education. Explaining a physical fact via mathematics in order to make it understandable *is* a mathematical practice, and first of all a pedagogical practice. In particular, if I assume with De Regt and Dieks (2005) that understanding transcends the domain of individual psychology and is relative to scientific communities in a specific historical period (they call it the "meso-level in science"), the importance of the acquisition of skills should be take into account in a more complete analysis. As remarked by Jeremy Avigad (2008) in his discussion of the notion of understanding in mathematical proofs:

> We look to mathematics for understanding, we value theoretical developments for improving our understanding, and *we design our pedagogy to convey understanding to students*. Our mathematical practices are routinely evaluated in such terms. It is therefore

reasonable to ask just what understanding amounts to (Avigad, 2008, p. 449. My emphasis).

So mathematical education is directly linked to the concepts of understanding, mathematical explanation and to the intelligibility standard of visualizability. In this direction: transitions in the formulation of Euler's theorem in mathematical (and physical) textbooks could be very helpful in order to study mathematical explanation in our sense and the variation of "what is considered more understandable" from a pedagogical point of view.

In the next Section I will briefly give an outline of the theorem and the two different mathematical explanations for the physical phenomenon. In Section 3 I will claim that MEPPs in this particular case have dynamical character, while in Section 4 I will focus on visualizability, understanding and on the particular role of vector space theory. I will defend the epistemic relevance of a contextual notion of understanding and I will put in evidence a shift in the notion of visualizability for this particular case of explanation. The final section will contain my conclusions and some epistemological and educational perspective.

2. EULER'S THEOREM

2.1 Euler's Original mathematical formulation in E177

Euler's contributions to mechanics are numerous and of primary importance. Between them, the remarkable fact that Euler was the first to prove the existence of an instantaneous axis of rotation in the kinematics of rigid body motion. He obtained the result of the instantaneous axis of rotation for the first time in his paper E177 *Decouverte d'un nouveau principe de Mecanique.* In this work Euler utilizes previous results in order to study the general motion of a rigid body with a fixed point and deduce the changes in the position and the velocity distribution from the given forces acting on the body [3]. His enterprise in the dynamics of rigid body motion in space was stimulated by the problem of the rotation of the Earth around its axis (as to explain the precession of equinoxes). The introduction of the perpendicular rectangular frame of reference permits Euler to apply Newton's second law separately with respect to each of the coordinates. This was brought about by a kinematical result: the instantaneous axis of rotation.

In the section *Détermination du mouvement en général, dont un corps solide est susceptible, pendent que son centre de gravité demeure en repos*, in order to study the velocity distribution, Euler introduces a cartesian system fixed in absolute space and assumes that a point Z of the body with coordinates x, y, z has velocities P, Q, R in the direction of the axis. The components of the velocity P, Q, R are functions of x, y, z. Euler's final purpose is to found those functions. He considers another point z "infiniment proche du précédent Z", of coordinates $x + dx$, $y + dy$, $z + dz$ and velocities $P + dP$, $Q + dQ$, $R + dR$. After a mixed geometrical-analytical procedure

Euler is able to state that there are points, which have coordinates (*Cu*, *-Bu*, *Au*), that do not move during time *dt*. In other words, those points are on a straight line through the origin, which is called the instantaneous axis of rotation [4].

> ... tous les points du corps, qui sont contenus dans ces formules x=Cu, y=-Bu, z=Au demeureront en repos pendant le tems dt. Or tous ces points se trouvent dans une ligne droite, qui passe par le centre de gravité O; donc cette ligne droite demeurant immobile sera l'axe de rotation, autour duquel le corps tourne dans le présent instant (Euler, 1750. p. 95).

Euler also added a geometrical proof of the existence of the instantaneous axis of rotation, discussing the infinitesimal motion of a spherical surface with a fixed point. The geometrical argument provided by Euler legitimates his analytical argument and holds not only for the instantaneous case but also for the discrete case.

2.2 A Modern formulation in Linear Algebra

As originally proved by Euler, the theorem for rigid body motion states that: "The general displacement of a rigid body with one point fixed is a rotation about some axis". The motion of a rigid system in modern mechanics is described specifying at each instant the position of the points of the body with reference to a system of axis. To every point we associate a vector which belongs to an euclidean 3-dimensional space. The orientation of the rigid body in motion can be described at any instant by an orthogonal transformation, the elements of which may be expressed in terms of some suitable set of parameters. With the progression of time the orientation will change and the matrix of the transformation will evolve continuously from the identity transformation **A**(0)=**1** to the general matrix **A**(t). Here we assume that at time t = 0 the body axes (the axes fixes in the rigid body) are chosen coincident with the space axes (a system of axes parallel to the coordinate axes of external space). The assumption that the operation implied in the matrix **A** describing the physical motion of the rigid body is a rotation assures that one direction (the axis of rotation) remains unaffected in the operation and the same holds for the magnitude of the vectors. If we consider as the fixed point in the rotation the origin of the sets of axes (and not necessarily the center of mass of the object), the displacement of the rigid body involves no translation of the body axes and we can restate Euler's theorem in the following modern mathematical form: "Every matrix **A** in SO(3), with **A** different from $\mathbf{I}_3$, has an eigenvalue +1 with a 1-dimensional eigenspace" (Sernesi, 1993, p. 305).

A proof of the mathematical theorem in the form I have given involves the general concepts of matrix, vectors (in particular the more specialized concepts of eigenvalue and eigenvector), eigenspace, basis, orthogonality, bilinear forms (in particular the scalar product, which is a symmetric and non-degenerate bilinear form). All those concepts are included in linear algebra and their close interplay does not permit any easy separate analysis of the elements which are found in the proof structure of such a theorem. As observed by Israel Klein in his *History of Abstract Algebra*:

> Among the elementary concepts of linear algebra are linear equations, matrices, determinants, linear transformations, liner independence, dimension, bilinear forms, quadratic forms, and vector spaces. Since these concepts are closely interconnected, several usually appear in a given context (e.g. linear equations and matrices) and it is often impossible to disengage them (Klein, 2007, p. 79).

The modern proof of the algebrical formulation is constructed into the general framework of linear algebra and the particular framework of euclidean 3-dimensional vector space $\mathbf{R}^3$. Clearly, the proof's outcome is to show the existence of the eigenvalue λ=1 [5]. If we do not consider the concept of group, and we focus on the general concept of vector space, we could analyse the explanatory structure and make some relevant remarks.

3. SHIFT IN MATHEMATICAL EXPLANATION

It is clear that Euler did not have at disposition the modern concept of vector and vector space. But, as we can see from his papers, he did have the basic idea of geometrical transformation (point-to-point association in space and not transformation from physical magnitude to geometrical magnitude), which was central to his analysis. Differently from Euler's original argument, in which the mathematical explanation is given by a mixed geometrical-analytical argument by means of a geometrical space (and *via* a geometrical intuition [6]), the modern explanation of the existence of an instantaneous axis of rotation is given in the framework of linear algebra. Having the particular structure of euclidean 3-dimensional vector space is essential to Euler's theorem because only the mathematical properties of a real vector space equipped with scalar product permit to "map" the properties of the kinematical system (angles, distances, orthogonality condition) into the algebraic structure.

In a recent paper Gingras (2001) has underlined how the shift in explanation and the "disparition of substances into the acid of mathematics" are an epistemic and an ontological effect of the process of mathematization started with Newton. As a consequence of an historical process concepts like determinant, matrix, orthogonality or transformation are now included in the mathematical apparatus of linear algebra and we could profit of their interplay without exit from this framework. In other words: in the modern algebrical proof the geometrical part is already "included'" in the structure of vector space and we do not need a geometrical argument [7]. It is very interesting to observe that Peano himself, in his *Analisi della teoria dei vettori*, remarked:

> Thus the theory of vectors appears to be developed without presupposing any previous geometrical study. And since, by means of this theory, all of geometry can be treated,

> there results thereby the theoretical possibility of substituting the theory of vectors for elementary geometry itself (Peano, 1898, p. 513).

After having proved the dynamical character of mathematical explanation (i.g. the mathematics is essential to both the two explanations but it *changes*), in the following Section I will use De Regt & Dieks's criteria for understanding and intelligibility in order to show how the theory of vector space offers a new conceptual tool of intelligibility and understanding.

4. UNDERSTANDING AND VISUALIZABILITY IN MEPP.

If I admit (and I do!) with De Regt & Dieks (2005) that visualizability constitutes a context-dependent standard of intelligibility, and only intelligible theories can provide an understanding of phenomena, then I can look at the shift between our two MEPPs in a more fruitful and interesting way. But, first of all, it is necessary to give a definition of visualizability, intelligibility and understanding.

As showed by De Regt (2001) being a spacetime theory is a necessary but not sufficient condition for visualizability. It might be objected here that I deal with mathematical entities and the term "spacetime" is very dangerous and misleading. Fortunately, I am referring to MEPPs and for my particular test case the conditions of necessity and sufficiency for visualizability are both fulfilled (Euler's geometrical framework and the framework of vector space theory both make the physical phenomenon *visualizable* in space -as a vector- at a particular time *t*, as could be seen from the diagrams we find in a common textbook of mechanics or mathematics). We can say that geometrical space in Euler and the modern concept of vector space *map* the physical space into a structure (a geometrical and a mathematical structure). In the case of vector space this mapping consists in an explicitly assumed isomorphism between the physical space and the 3-dimensional Euclidean space.

De Regt & Dieks (2005) propose two criteria for understanding and intelligibility: CUP (Criterion for Understanding Phenomena) and CIT (Criterion for the Intelligibility of Theories).

> CUP: A Phenomenon P can be understood if a theory T of P exists that is intelligible (and meets the usual logical, methodological and empirical requirements).

The necessary connection between visualizability and understanding is made by De Regt through the Criterion for the Intelligibility:

> CIT: A Scientific Theory T is intelligible for scientists (in context C) if they can recognize qualitatively characteristic consequences of T without performing exact calculations.

In the previous Criterion I substitute "Mathematical Theorem" for "Scientific Theory" and I assume the uniformity of the CIT in both cases (with some differences that should be discussed). But how do we "recognize qualitatively characteristic consequences of T without performing exact calculations"? A possible answer: through conceptual tools. In a particular historical or methodological context we have at disposition some conceptual tools and visualizability *could be* one of them [8]. In other words: visualizability is a conceptual context-dependent tool, i. g. a conceptual contingent tool which depends from the skill of the scientific-mathematical community and which is present during a precise historical period, and it *could* permit the intelligibility of a theory making possible the circumvention of the calculatory stage and the jump to the conclusion. So it is clear that also intelligibility is context-dependent. Naturally, as remarked by De Regt (2001), visualizability is not a necessary condition for intelligibility. Often other conceptual tools as abstract reasoning or familiarity could lead scientists and mathematicians to intelligibility as an immediate conclusion (see De Regt & Dieks, p. 156, for examples). Mathematical practice and theoretical physics are full of situations like this.

In Euler the tool of visualization is perfectly applicable in the classical geometrical framework (I call it Euclidean Geometrical Theory): point-to-point association and geometrical considerations offer the idea (a *visual idea*) of what is happening to the mechanical system in motion. The instantaneous axis of rotation could always be visualized in spacetime, and its existence could be established through a geometrical-intuitive reasoning [9]. In the modern explanation given in the framework of abstract algebra it might seem that this "chance" of intellegibility has been lost, but a deeper look shows that this is not completely true. The concept of 3-dimensional Euclidean vector space offers two new ways for obtaining the intelligibility (in line with CIT). Reading the modern formulation of Euler's theorem a mathematician or a student could affirm "Yes, I see the eigenvalue +1", just by looking at the formulation of the theorem in the matrix formalism. This is associated with the conceptual tool of familiarity, or abstract reasoning, and is related to a previous learning of matrix theory or other mathematical abilities. Instead of this approach, one can reach the same direct conclusion just by considering some general results in matrix theory and *visualizing* the eigenvector (the instantaneous axis) in the diagram [10]. The latter can be considered a new conceptual tool leading to the fulfilment of CIT. Naturally, the structure of nxn matrices with entries from **R** and the structure of homomorphisms of a 3-dimensional space (over **R**) into itself are isomorphic. From the last considerations is clear that visualizability still plays a very important role in understanding and in developing a fruitful strategy of mathematical education.

5. CONCLUSIONS AND PERSPECTIVES

MEPPs are context-dependent and have dynamical character. In particular, *via* a contextual approach to understanding, it is possible to recognize that the framework of linear algebra has defined new standards (or tools) for intelligibility which legitimate an explanation as "a good explanation" (an explanation which produces understanding). The understanding in this context is a payoff that directly comes from the availability of those conceptual tools. As I have showed, in the modern formulation the understanding of the mathematical explanation for the existence of an instantaneous axis of rotation is obtained through *a double route* (visualization and abstract reasoning). I claim that this result might be very helpful in mathematical education and could offer a possible answer to Avigad's question "How do we design our pedagogy to convey understanding to students?" for the specific case discussed. A new interesting direction, as showed by Marcus Giaquinto in his studies on the epistemic function of visualization in mathematics (Giaquinto, 2005), could emerge from an analysis of visualization as a powerful educational tool in the context of discovery [11].

A better comprehension of mathematical explanation could profit from the historical study of the interplay between the proof structure of the theorem and the system of concepts that characterizes the explanatory structure. If a change in one of them influences the other, it could be interesting to study different formulations of Euler's theorem in textbooks in order to see how the mathematical explanation has been offered during this period and how it has changed in mathematical education. Naturally, the epistemological analysis of this paper opens the way to the more general question of how introduce proofs in classrooms and how concepts like explanation, understanding and visualizability should be taken into account in mathematical education.

NOTES

1. For shortness, from now on, I will refer to Mathematical Explanation of Physical Phenomena with the term MEPP.

2. For a panoramic of this field and the very interesting discussion of this point, including how computer graphics has helped to recognize mathematical structures such as Julia sets which would have been impossible to recognize analytically, see Mancosu (2005).

3. For a more precise reconstruction of Euler's argument in Euler (1750) see the paper "What we can learn about mathematical explanations from the history of mathematics" I've presented at *Novembertagung* Conference, in Denmark, 5-9 November 2008.

4. Euler does not use the word "instantaneous axis". He refers to it simply as "axe de rotation".

5. For a proof of the theorem see Sernesi (1993, p. 306).

6. The importance of the geometrical intuition in Euler emerges from the geometrical proofs he adds after his analytical arguments. The geometrical argument defines and legitimates the analytical procedure and is essential to the mathematical explanation of the existence of the axis.

7. Vector spaces firstly appear in their axiomatic form in Peano (1888).

8. Evidently, the intelligibility standard or tool of "casual connection" is of no interest in our discussion.

9. See Euler's geometrical argument or a modern geometrical argument (Whittaker, 1904, p. 2).

10. Here I am not claiming that the geometrical interpretation of matrices and eigenvectors is intrinsic in their definitions. I am assuming that under a particular "reading" (in our case Euler's theorem in kinematics of rigid body motion), a subset of vectors of the vector space considered (the subset containing the instantaneous axis) has a geometrical representation in a diagram at time *t* (or a representation in a computer graphic simulation). A very good example of a case in which a precise situation is visualizable in the context of Vector Space Theory has been given by Artin (1957) and is discussed in Tappenden (2005).

11. For simple and interesting cases in which a case of visualization could provide the discovery of a theorem see Giaquinto (2005) or, in a different flavour, the famous Lakatos (1978).

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PROVING AS A RATIONAL BEHAVIOUR: HABERMAS' THEORY OF RATIONALITY AS A COMPREHENSIVE FRAME FOR RESEARCH ON THE TEACHING AND LEARNING OF PROOF

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In this paper we draw from Habermas' theory of rationality a model for rationality in proving that we propose as suitable to investigate the teaching and learning of proof and generate new research developments. At first, we discuss our conception of the proving process, where cognitive and cultural aspects are shown to play a crucial role, and we present our adaptation of Habermas' model as a way of taking into account both cognitive and cultural aspects. The model is shown to be useful in the discussion of some examples at tertiary level; finally, drawing from the analysis of the examples, we indicate some research questions (formulated in terms of the model of Habermas) that we feel worth to be explored.

Key-words: proof and proving, rational behaviour, Habermas, tertiary level

INTRODUCTION

The aim of our paper is to contribute to the debate on theoretical frameworks suitable to take into consideration the complex nature of the teaching and learning of proof.

When planning the teaching of theorems and mathematical proof and when analyzing students' difficulties in approaching them, we have at disposal several theoretical tools coming from epistemology, history of mathematics, psychology, and didactics of mathematics. In order to build a comprehensive framework for proof and its teaching and learning, encompassing the epistemological, psychological and didactical dimensions, we think that at first it is necessary to consider proof as a crucial component of mathematics and to look at mathematics from a cultural perspective. The definition of culture by Hatano & Wertsch (2001) suggests to consider mathematics as a multifaceted culture evolving through the history, which includes different kinds of activities and different levels of awareness, explicitness and voluntary use of notions, thus different levels of "scientific" mastery, according to the Vygotskian distinction between common knowledge and scientific knowledge (for further developments about mathematics as a culture, see Morselli, 2007). Within this cultural perspective we can situate the "culture of theorems" as the complex system of conscious systematic knowledge, activities and communication rules that refer to the processes of conjecturing and proving as well as to their final products. Consequently, we can describe the approach to theorems and proving as a process of scientific "enculturation" consisting in the development of a special kind of rational behaviour, characterized by the conscious mastery of the epistemic aspects of theorems (Mariotti et al., 1997; Balacheff, 1982) and by the intentional construction and control of the process that produces the proof, within a communication context with its shared rules. From these considerations we can draw a link between the approach to theorems as a process of “scientific enculturation” and the three components of Habermas’ “rational behaviour” (the epistemic, the teleological and the communicative rationalities), as we will show in the subsequent section.

Another entry into the same line of thought derives from the process - product character of proving and proof. Balacheff (1982) points out that the teaching of proofs and theorems should have the double aim of making students understand what a proof is and learn to produce it. Accordingly, we think that, in mathematics education, proof should be treated considering both the *object* aspect (a product that must meet the epistemic and communicative requirements established in today mathematics - or in school mathematics) and the *process* aspect (a special case of problem solving: a process intentionally aimed at a proof as product). Here again we can identify potential links with Habermas’ elaboration about rationality in discursive practices.

PROVING AS A RATIONAL BEHAVIOUR

Habermas (2003, ch. 2) distinguishes three inter-related components of a rational behaviour in discursive practices: the *epistemic* component (inherent in the control of the propositions and their enchaining), the *teleological* component (inherent in the conscious choice of tools to achieve the goal of the activity) and the *communicative* component (inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas’ elaboration, in the discursive practice of proving we can identify: an *epistemic* aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of “theorem” by Mariotti & al. (1997) as the system consisting of a statement, a proof which is derived according to shared inference rules from axioms and other theorems, and a reference theory); a *teleological* aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product; a *communicative* aspect, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture.

Our point is that considering proof and proving according to Habermas’ construct may provide the researcher with a comprehensive frame, within which to situate a lot of research work performed in the last two decades, to analyze students’ difficulties concerning theorems and proofs (see the four examples in the next Section) and to discuss some related relevant issues and possible implications for the teaching of theorems and proof (see the last Section).

If we are interested in the epistemic rationality side, i.e. in the analysis of proofs and theorems as objects, mathematics education literature offers some historical analyses (like Arsac, 1988) and surveys of epistemological perspectives (like Arzarello, 2007): they help to understand how theorems and proofs have been originated and have been considered in different historical periods and how, even in the last decades, there is no shared agreement about what makes proof a "mathematical proof" (cf. Habermas' comment about the historically and socially situated character of epistemic rationality). Concerning the ways mathematical proof and theorems are (or should be) introduced in school as "objects", several results and perspectives have been produced, according to different epistemological perspectives and focus of analyses. In particular, De Villiers (1990), Hanna (1990), Hanna & Barbeau (2008) discuss the functions that mathematical proofs and theorems play within mathematics and advocate that the same functions should be highlighted when presenting proof in the classroom, in order to motivate students to proof and allow them to understand its importance. By referring proof to the model of formal derivation, Duval (2007) focuses on the distance between mathematical proof and ordinary argumentation; he also considers how to make students aware of that distance and able to manage the construction and control of a deductive chain. Harel (2008) uses the DNR construct to frame the classification of students' proof schemes (we may note that they concern proof as a final product). We note that, in terms of Habermas' components of rationality, Harel's ritual and non-referential symbolic proof schemes may be attributed to the dominance of the communicative aspect, with lacks inherent in the epistemic component (cf. Harel's N, "intellectual Necessity").

Concerning the proving process, some analyses of its relationships with arguing and conjecturing suggest possible ways to enable students to manage the teleological rationality. In particular, Boero, Douek & Ferrari (2008) focus on the existence of common features ("cognitive unity") between arguing, on one side, and proving processes on the other, and present some activities (from grade I on), based on those commonalities, that may prepare students to develop effective proving processes. Research on abductive processes in conjecturing and proving (Cifarelli, 1999; Pedemonte, 2007) and the construct of "abductive system" (Ferrando, 2006) take into account some aspects of the creative nature of conjecturing and proving processes and the need of suitable educational choices to promote creativity. Boero, Garuti & Lemut (2007) suggest the possibility of smoothing the school approach to mathematical proof through unified tasks of conjecturing and proving for suitable theorems (those for which the same arguments produced in the conjecturing phase can be used in the proving phase). However Pedemonte (2007) shows how in some cases of "cognitive unity", students meet difficulties inherent in the lack of "structural continuity" (when they have to move from creative ways of finding good reasons for the validity of a statement, to their organization in a deductive chain and an acceptable proof): her study suggests to consider the relationships between teleological, epistemic and communicative rationality (see the last Section).

SOME EXAMPLES

Morselli (2007) investigated the conjecturing and proving processes carried out by different groups of university students (7 first year and 11 third year mathematics students, 29 third year students preparing to become primary school teachers). The students were given the following problem: *What can you tell about the divisors of two consecutive numbers? Motivate your answer in general*. Different proofs can be carried out at different mathematical levels (by exploiting divisibility, or properties of the remainder, or algebraic tools). The students worked out the problem individually, writing down their process of solution (including all the attempts done); afterwards, students were asked to reconstruct their process and comment it. The *a posteriori* interviews were audio-recorded. In (Morselli, 2007) several examples of individual solutions and related interviews are provided, and in particular it is shown how students' failures or mistakes were due to lacks in some aspects of rationality and/or the dominance of one aspect over the others.

For the present paper, we selected four examples. At first, we present two very similar cases, concerning students that are preparing to teach in primary school, and we show how the model of rationality may help to single out important differences between the two students, as well as different needs in terms of intervention. Afterwards, we present two cases concerning university students in Mathematics: the first one is a case of success, while the second one is a case of failure. These two cases were analyzed in (Morselli, 2006) with a special focus on their use of examples. Here we discuss those proving processes by means of Habermas' model.

The four examples have the double aim of illustrating how Habermas' construct works as a tool for in-depth analysis, and introducing a discussion that will suggest further research developments.

Example 1: Monica

Monica considers two couples of numbers: 14, 15 and 24, 25. By listing the divisors, she discovers that "Two consecutive numbers are odd and even, hence only the even number will be divided by 2". Afterwards, she lists the divisors of 6 and 7 and writes: "Even numbers may have both odd and even divisors". After a check on 19 and 20, she writes the discovered property, followed by its proof:

> Property: two consecutive numbers have only one common divisor, the number 1. In order to prove it, I can start saying that two consecutive numbers cannot have common divisors that are even, since odd numbers certainly cannot be divided by an even number. They also cannot have common divisors different from 1, because between the two numbers there is only one unit; if a number is divisible by 3, the next number that is divisible by 3 will be greater by 3 units, and not by only one unit. Since 3 is the first odd number after 1, there are no other numbers that can work as divisors of two consecutive numbers.

Monica carries out a reasoning intentionally aimed (teleological rationality): first, at the production of a good conjecture; then, at its proof. Proof steps are justified one by one (epistemic aspect) and communicated with appropriate technical expressions (communicative aspect). The only lack in terms of rationality concerns the short-cut in the last part of the proof: Monica realizes that something similar to what happens with 3 (the next multiple is "greater by three units") shall happen a fortiori with the other odd numbers that are bigger than 3 ("Since 3 is the first odd number after 1"), but she does not make it explicit. Her awareness (cf. epistemic rationality) is not communicated in the due, explicit mathematical form (lack of communicative rationality). Monica's *a posteriori* comments on her text confirm the analysis:

> Monica: (...) and then I have thought that 3 was the first odd number after 1 and so if 3 does not enter there, also the bigger ones do not enter there [from the previous text, we know that "there" means: between two consecutive numbers on the number line].
>
> Interviewer: to make more general what you said with 3, what would you write now?
>
> Monica: ehm... I have tried to go beyond the specific case of 3, but I do not know if I have succeeded in it.

Example 2: Caterina

> Starting from the fact that two consecutive numbers are always one odd and one even, we may conclude that the two numbers cannot be both divided by an even number. Afterwards, we focus on odd divisors; we start from 1, and we know that all numbers may be divided by 1; the second one is 3. We have two consecutive numbers, then the difference between them is 1, then they will not be multiples of 3, since it will be impossible to divide both of them by a number bigger than 1.

Caterina is able to justify all the explicit steps of her reasoning (epistemic rationality), she develops a goal-oriented reasoning (teleological rationality) and illustrates her process with appropriate technical expressions (communicative rationality). Differently from Monica, in spite of a good intuition there is a lack in her reasoning: divisors greater than 3 are not considered. A posteriori, after having seen also the production of her colleagues, Caterina comments:

> My reasoning is not mistaken: indeed, I reach the conclusion giving a general explanation, saying that, since there is no more than one unit between the two numbers, the only common divisor is 1. Nevertheless, I can not create a mathematical rule. Observing the other solutions, I think that the correct rule is the following: along the number line we note that a multiple of 2 occurs every two numbers, a multiple of 3 occurs every three numbers, hence a multiple of N occurs every N numbers. Then, two consecutive numbers have only 1 as common divisor.

From the objective point of view of epistemic rationality, Caterina's argument was not complete, and in her comment she reveals not to be aware of it. From her subjective point of view, Caterina is convinced to have found a cogent reason for the validity of the conjecture ("not mistaken reasoning", "general explanation"), thus to have achieved her goal (teleological rationality). Some colleagues' solutions induce her to reflect on the lack of a "mathematical rule"; however she doesn't seem to consider this lack as a lack in the reasoning, but as a lack in the mathematical communication.

Example 3: Sara

Sara (attending the third year of the university course in Mathematics), after having discovered the property by means of two numerical examples (*1-2, 2-3*), writes down:

> "Two consecutive numbers are "made up" of an even number, divisible by 2 (=2n, n∈N) and an odd number (*=2n+1*, *n∈N*). Let's suppose that *1* is not the only common divisor, that is *∃ k* such that *k/2n* and *k/2n+1*. *2n= ka*, *a∈N* → also in *ka* there must be the factor *2* → *k=2c* or *a=2d*; *2n+1= kb*, *b∈N* → since *k* is common, *k=2c*, or *b=2e*. But only the product of two odd numbers is an odd number → I could not finish for a matter of time."

Sara seems to be aware of the way a proof should be presented (communicative rationality), of the importance of algebra as a proving tool and of the usefulness of the proof by contradiction in a case like this (two important strategic choices concerning teleological rationality). In particular, in the *a posteriori* interview she tells that she felt comfortable with the method of proof by contradiction, due to the fact that the uniqueness of 1 as a common divisor had to be proven.

Even epistemic rationality works till the last part of her algebraic work, where she derives the incorrect conclusion that "*k=2c*, or *b=2e*". However Sara gets lost after a few manipulations. Why did it happen? It is possible that in this case the arguments successfully used in the conjecturing phase (based on the distinction between odd and even, and thus on divisibility by 2) were misleading when applied in the proving phase. Incidentally, here we see that in some cases cognitive unity may act as a burden, if not controlled. Indeed, Sara could have reached the proof easily by substituting *2n=ka* in the expression *2n+1=kb*, but she didn't take into consideration this strategy, she just focused on divisibility by two. Substituting *2n=ka* in the expression *2n+1=kb* would have required to move from the odd/even semantic-based argument to a pure algebraic manipulation, with a break in the continuity of the conjecturing and proving process. Probably, Sara got lost because, when orienting her proving process, she did not fully concentrate on the meaning of the expression "1 is not the only common divisor", being still focused on the odd-even dichotomy. Even her mistake (when she derived " *k=2c*, or *b=2e*" from the previous step) might have depended on her intention to get the absurd conclusion that 2n+1 would have been even (indeed she wrote: "But only the product of two odd numbers is an odd number"). Thus her failure might be interpreted in terms of one of her strategic choices not fitting with the aim of the proving process and not supported by a rigorous checking of inferences (i.e. in terms of a combined lack on the epistemic and teleological dimensions of rationality).

Example 4: Valentina

Valentina (attending the third year of the university course in Mathematics) chooses to carry out her exploration through an algebraic manipulation.

> Given *n∈N*, if it is divisible by *d∈N*, then the remainder of the division of *n* by *d* is *0*, that is to say *n mod d* is *0*, that is to say in Z_d *n=0*. When I consider *n+1*, reasoning in the

same way I realize that dividing by *d* I get remainder *1*, that is to say $n+1=1$ in Z_d $\forall d\neq 1$. Then, the only common divisor for *n* and *n+1* is *1*.

The exploration carried out by Valentina seems to be very useful: at the same time Valentina discovers the property and proves it, since the reasoning is already carried out in general terms. In the subsequent excerpt from the *a posteriori* interview, Valentina describes her process of conjecturing and proving. Valentina, being aware of the potentialities and limits of numerical examples, chooses to use algebra also in the exploration phase. We may say that the epistemic dimension (awareness of the limits of numerical examples) supports the teleological one (choice of algebra in the exploratory phase).

Interviewer: Try to explain to a secondary school student how to find the property.

Valentina: I think that… beh, I would start reasoning on data, on the hypotheses, and trying to see links between them, seeing what happens in various cases?

Interviewer: do you mean using numerical examples?

Valentina: maybe, even if this could be dangerous because induction does not always works, I mean, if we have limited cases, it is not a good method, it could even be absolutely wrong. But one could start from them; afterwards of course it is necessary to prove it in general… […] and just consider the hypothesis and try and think about them, from a general point of view, just…non numerical, but *n*, *n+1*, what they mean, and try exactly to think about them, what this data mean.

Let us come back to Valentina's production. After the first phase, in which Valentina discovers and proves the property at the same time, Valentina writes down: "That were my fist ideas. Now I try to write them down in a better way". This sentence leads to a phase of systematization of the final product.

Given n∈ N, n and n+1 have only one common divisor, that is 1. In fact, ∀ d∈N such that d/n, d≠1, (n)=(0) in Z_d, while (n+1)=(1) in Z_d because (n+1)=(n)+(1)=(0)+(1)=(1), hence d~/n+1. From the other side, ∀ p∈N such that p/n+1 and p≠1 I have that (n+1)=(0) in Z_p and that (n)=(n+1-1)=(n+1)-(1)=(0)-(1)=(-1), hence p~/n. On the contrary, 1/n and 1/(n+1) because 1 divides any natural number.

In the subsequent excerpt from the *a posteriori* interview, Valentina shows to put a great care both in the process and in the construction of the final product.

Interviewer: ok. May I ask you why did you do a second part, in which you systematized what you wrote in the first part?

Valentina: the first part was… I gave the idea, I started to write down, in a sort of draft, in order to make my ideas clear to myself, in order to formalize what I had in my mind. Afterwards, I tried to write in a more formal way, because the first part was really… writing down ideas, while in the second part I tried to write in a more "mathematical" way, in clearer way.

Interviewer: what do you mean by "more mathematical way"?

Valentina: ehm… maybe using less words, trying to be more synthetic, and trying to use a mathematical language, then with more symbolic notation, rather than words.

Interviewer: ok. But actually, as concerns the mathematical content…

Valentina: it is the same. It is more or less the same. Yes, yes.

We may note that Valentina is able to describe the features that, according to her, a mathematical proof should have. Nevertheless, Valentina is aware that the first part of her production is already acceptable, even if written in a less appropriate way. We may say that Valentina is able to manage the crucial dialectic between epistemic and communicative dimension: the second part is an amendment from the communicative point of view, but Valentina is fully aware of the fact that the communication is subordinated to the epistemic dimension, that is to say to the validity of the produced arguments.

DISCUSSION: TOWARDS FURTHER DEVELOPMENTS

The analysis of some examples had the double aim of showing the viability and usefulness of the model of Habermas in the special case of conjecturing and proving, and of suggesting new research questions, in terms of the model of Habermas.

As concerns the first aim, we have seen how success and failure may be read in terms of different intertwinings between the three components of rationality, or dominance, or lack on one of them. We may add that in the case of Valentina the communicative component is strictly depending on the epistemic one; furthermore, the teleological component intertwines with the epistemic one (choice and justification of the arguments) and with the communicative one (other readers will check the production). More generally the previous analyses suggest the opportunity of a closer investigation into the relationships between epistemic rationality, communicative rationality and teleological rationality in the case of proof and proving. Concerning this issue we note that in the historical development of mathematics, subjective evidence (or even mathematicians' shared opinion of evidence) revealed to be fallacious in some cases, when new, more compelling communication rules obliged mathematicians to make some steps of reasoning (in particular, those concerning definitions: see Lakatos, 1976) fully explicit.

From the educational point of view, while it is easy (for instance, by comparison with other solutions) to help Monica to make her reasoning more explicit (according to her need, as emerged from her comments), the intervention on Caterina is much more delicate: how to make her aware that the "mathematical rule" is not only a matter of conventional, more complete communication, but also a matter of objective, cogent arguing involving the goal to achieve (an exhaustive argument)? And how to exploit texts that are complete (communicative aspect) in order to develop the need of an exhaustive argument (epistemic aspect), but at the same how to avoid that the necessities inherent in the communicative aspect prevail over the epistemic aspect (cf. Harel's "ritual proof schemes")? A direction for productive educational developments might consist in the elaboration of a suitable meta-mathematical discourse (see Morselli, 2007) for students (including an appropriate vocabulary), as well as in the choice of suitable tasks that reveal how intuitive evidence not developed into an explicit, detailed justification sometimes results in fallacious conclusions.

These considerations raise another problem: Habermas' construct offers only the possibility to evaluate a production process and its written or oral products, while in mathematics education we need also to consider a long term "enculturation" process. We are working now on the articulation between a cultural perspective to frame this process (see Morselli, 2007) and tools of analysis derived from Habermas' elaboration on rationality. Indeed, it is within the cultural perspective outlined in the introduction that we think possible to deal with the approach to theorems and proving in school as a process of scientific "enculturation" consisting in the development of a special kind of rational behaviour, the one derived from Habermas, that is presented in this paper. We are trying to refine the Vygotskian common concepts - scientific concepts dialectics in the case of theorems and proofs in order to get a frame where to situate the long term planning of the school approach to the culture of theorems. Habermas' construct contributes to it by suggesting three interrelated dimensions along which to develop students' skills in proving and students' (and teachers') awareness about crucial features of proving and proofs. The educational challenge consists in leading students to move from the ordinary argumentative practices of validation of statements in different domains to the highly sophisticated and culturally situated management of the components of a rational behaviour in the specific case of proving.

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TEACHERS' VIEWS ON THE ROLE OF VISUALISATION AND DIDACTICAL INTENTIONS REGARDING PROOF

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In this paper we explore secondary teachers' views on the role of visualisation in the justification of a claim in the mathematics classroom and how these views could influence instruction. We engaged 91 teachers with tasks that invited them to: reflect on/solve a mathematical problem; examine flawed (fictional) student solutions; and, describe, in writing, feedback to students. Eleven teachers were also interviewed. Here we draw on the interviews and the responses to one Task (which involved recognising a line as a tangent to a curve at an inflection point) of two teachers. We do so in order to explore potential influences on the didactical contract regarding proof that these teachers are likely to offer their students. One such influence is the clarity and stability of their beliefs about the role of visualisation.

Key Words: teacher beliefs, proof, visualisation, tangents, didactical contract

INTRODUCTION

'The emphasis that teachers place on justification and proof no doubt plays an important role in shaping students' 'proof schemes''[1] (Harel & Sowder, 2007, p827). The not very extensive research in this area (p824) shows that this emphasis is insufficient both in terms of extent and in terms of quality. Internationally in most educational settings – even those with an official curricular emphasis on proof – little instructional time is dedicated to proof construction and appreciation (p828). Furthermore teachers' own proof schemes are often predominantly empirical and teachers do not always seem to understand important roles of proof other than verification (p836). For example, in Knuth's (e.g. 2002) study of practising secondary mathematics teachers, while all teachers acknowledged the verification role of proof, they rarely talked about its explanatory role. With regard to their proof schemes many of the interviewed teachers: felt compelled to check a statement on several examples even though they had just completed a formal proof; considered several of given non-proofs as proofs; and, accepted the proof of the converse of a statement as proof of the statement; and, found arguments based on examples or visual representations to be most convincing.

One of the aims of the study we report in this paper is to explore the relationship between teachers' pedagogical and epistemological beliefs about proof and their intended pedagogical practice (e.g. Cooney et al, 1998; Leder et al, 2002). Here we report some findings that relate to their beliefs about the role of visualisation.

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[1] Harel & Sowder's (1998) term which describes an individual's and a community's perception of proof. They distinguish between external conviction (authoritarian, ritual, non-referential symbolic), empirical (inductive, perceptual) and deductive (transformational, axiomatic) proof schemes.

In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (e.g. Mancosu et al, 2005), not least because developments in IT have expanded this potential so greatly. Central to this debate is whether, how and to what extent, visual representation can be used not only as evidence and means of insight for a mathematical statement but also as part of its justification (Hanna & Sidoli, 2007). For example, Giaquinto (2007) argues that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Visualisation has gained analogous visibility within mathematics education. Its richness, the many different roles it can play in the learning and teaching of mathematics – as well as its limitations – are increasingly being written about (e.g. Arcavi, 2003). These works address a diversity of issues, including: mathematicians' perceptions and use of visualisation; students' seeming reluctance to engage (and difficulty) with visualisation; etc. (Presmeg, 2006). Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several (e.g. Arcavi, ibid.) also recommend caution with regard to 'the 'panacea' view that mental imagery only benefits the learning process' (Aspinwall et al, 1997, p315).

One of the aims of the study we report in this paper is to contribute to the above debate as outlined in the work of Presmeg, Arcavi and others through exploring secondary mathematics teachers' beliefs about the role of visualisation as evident in the reasoning and feedback they present to students. The specific part of the debate our study aims to contribute to concerns the relationship between these beliefs and teachers' intended pedagogical practice. Our particular interest is in the potential influences on the didactical contract (Brousseau, 1997) that teachers offer their students with regard to the role of visualisation. One such potential influence is the clarity and stability of teachers' belief systems (Leatham, 2006). Below we briefly introduce the study.

THE STUDY AND THE TANGENT TASK

The data we draw on in this paper originate in a study, currently in progress in Greece and in the UK, in which we invite teachers to engage with mathematically/pedagogically specific situations which have the following characteristics: they are hypothetical but likely to occur in practice and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks we ask teachers to engage with is as follows – see a more elaborate description of the theoretical origins of this type of task in (Biza et al, 2007): reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s).

In what follows we focus on one of the tasks (Fig. 1) we have used in the course of the study. The Task was one of the questions in a written examination taken by candidates for a Masters in Mathematics Education programme. Ninety-one candidates (of a total 105) were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours.

Year 12 students, specialising in mathematics, were given the following exercise:
'Examine whether the line with equation $y = 2$ is tangent to the graph of function f, where $f(x) = 3x^3 + 2$.'
Two students responded as follows:

Student A

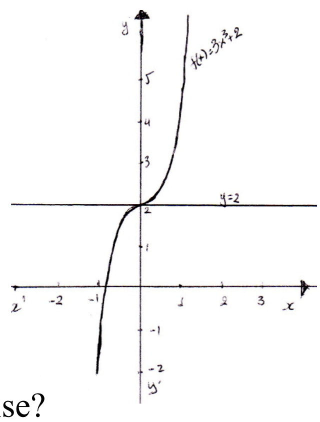
'I will find the common points between the line and the graph solving the system:

$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.
The line is tangent of the graph at point A because they have only one common point (which is A).'

Student B

'The line is not tangent to the graph because, even though they have one common point, the line cuts across the graph, as we can see in the figure.'



a. In your view what is the aim of the above exercise?
b. How do you interpret the choices made by each of the students in their responses above?
c. What feedback would you give to each of the students above with regard to their response to the exercise?

Figure 1: The Task

The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers' responses with regard to the following: perceptions of the *aims* of the mathematical exercise in the Task; mathematical *correctness*; *interpretation/evaluation* of the two student responses included in the Task; *feedback* to the two students. Adjacent to these columns there was a column for commenting on the approach the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. On the basis of this first-level analysis we selected 11 of the participating teachers for interview. Their individual interview schedules were tailored to the analysis of their written responses and, mostly, on questions we had noted in the last column of the spreadsheet. Interviews lasted approximately 45 minutes and were audio recorded.

The mathematical problem within the Task in Fig. 1 aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (e.g. Biza, Christou & Zachariades, 2008; Castela, 1995) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point *A* the problem provides an opportunity to investigate the two beliefs about tangency mentioned above – similarly to the way Tsamir et al (2006) explore teachers' images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

With regard to the Greek curricular context, in which the study is carried out, the Year 12 students (age 17/18) mentioned in the Task have encountered the tangent to the circle in Year 10 in Euclidean Geometry and the tangent lines of conics in Analytic Geometry in Year 11. In Year 12, they have been introduced to the tangent line to a function graph as a line with a slope equal to the derivative of the corresponding function at the point of tangency. Although in Years 11 and 12 the tangent is introduced as the limiting position of secant lines, this definition is rarely used in problems and applications. The students' mathematics 'specialisation' mentioned in the Task refers to the students' choice of mathematics as one of the curriculum subjects for more extensive study in Years 11 and 12.

The discussion we present in this paper is based on a theme that emerged from the first-level data analysis and was explored further in the interviews: the teachers' *beliefs about the role of visualisation in mathematics (epistemological) and in their students' learning (pedagogical).* This theme emerged largely from our observation that, in their scripts, the majority of the teachers distinguished between (and often juxtaposed) Student A's *algebraic* approach and Student B's *graphical* approach. Most of these teachers included in their comments an evaluative statement regarding the sufficiency/acceptability of one or both approaches. And often they referred explicitly to their beliefs about, for example, the **sufficiency/acceptability of the graphical approach**; or about the **role visual thinking may play in their students' learning**. The teachers' responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their **own perceptions of tangents** and their own views as to whether the line in the Task must be accepted as a tangent or not.

For example, with regard to the teachers' evaluation/interpretation of Student B's solution and feedback to Student B we scrutinised the scripts and designed the interviews with reference to questions such as: does the teacher turn the student away from the graphical approach (which may have led the student to an incorrect claim) and towards an algebraic solution in order to help the student change their mind about whether the line is a tangent or not? Does the teacher compare and contrast the algebraic solution to Student B's solution or do they proceed directly to the presentation of an algebraic solution? What types of examples/counterexamples, if any, do they employ in this process? What is the teacher's position towards Student B's grounding their claim on the graph and, generally, towards the validity of graphical argumentation as proof? Etc.. We presented a preliminary analysis of the above in (Biza, Nardi & Zachariades, 2008). This analysis suggested that there was substantial variation amongst the participating teachers in terms of the stability and clarity of their beliefs about the role of visualisation (epistemological and pedagogical). In what follows we present evidence from the scripts and interviews of two teachers, Spyros and Anna[2], whose cases exemplify this variation. Of particular interest in the accounts that follow is the interplay between the teachers' beliefs and their (stated) pedagogical practice. The data is translated from Greek.

SPYROS

Spyros has about fifteen years of teaching experience in secondary education. In his written response to the Task he described what led Student A and Student B to their respective answers. His feedback to the students was brief and stated rather generally. He emphasised the significance of mathematical definitions (in this case; the definition of tangent) and juxtaposed students' understanding and use of the definition with what he called 'intuitive' perception of the concept. He did not refer to any specific procedure through which the students could have determined whether the line is a tangent or not. At the same time he focused almost entirely, but rather generally, on the conceptual understanding of the definition and its 'history' in mathematics. We invited him to the interview in order to explore further his references to the 'history' of the concept and elaborate his feedback to the students.

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[2] We note that Spyros is one of the 38 (out of 91) teachers who rejected Student B's claim that the line is not a tangent. Anna is one of the 25 teachers who agreed with Student B's claim. There was some evidence of support for Student B's claim in the scripts of another 18 teachers and there were also 9 blank or half-completed scripts.

During the interview he stated that he had not thought about the relationship between the circle tangent and the tangent to a curve. He recognised that Student A had regarded having a unique common point as a sufficient condition for tangency and stressed that this condition is neither sufficient nor necessary. He also described counter-examples that could help Student A reconstruct their image of a tangent line.

While discussing Student B's response we asked him to elaborate on whether he would accept an argument based on a graph. His answer was firm: 'No, first of all it is not an adequate answer in exams'. (We note that in the Year 12 examination, which is also a university admission exam, there is a requirement for formal proof). We asked him to let aside the examination requirements for a moment and consider whether an argument based on a graph would be adequate mathematically. He replied: 'Mathematically, in the classroom, I would welcome it at lesson-level and I would analyse it and praise it, but not in a test'. Asked to elaborate he says: 'Through [the graph-based argument] I would try to lead the discussion towards a normal proof…with the definition, the slope, the derivative etc.'. Asked to justify he says:

> This is what we, mathematicians, have learnt so far. To ask for precision. For axiomatic… we have this axiomatic principle in our minds. Whatever I say I prove on the basis of axioms, on the basis of theorems, on the basis…. And this is what is required in the exams. And we are supposed to prepare the students for the exams.

In the above, Spyros's statement is clear: while he cannot accept a graph-based argument as proof, he recognises graph-based argumentation as part of the learning trajectory towards the construction of proof. He seems to approach visual argumentation from three different and interconnected perspectives: the *restrictions of the current educational setting*, in this case the Year 12 examination; the *epistemological constraints* with regard to what makes an argument a proof within the mathematical community; and, finally, the *pedagogical role* of visual argumentation as a means towards the construction of formal mathematical knowledge.

These three perspectives reflect three roles that a mathematics teacher needs to balance: educator (responsible for facilitating students' mathematical learning), mathematician (accountable for introducing the normal practices of the mathematical community) and professional (responsible for preparing candidates for one of the most important examinations of their student career). Spyros' awareness of these roles, and their delicate interplay, is evidence of the clear and stable didactical contract he appears to be able to offer to his students. Below we discuss a rather different case.

ANNA

Anna is a recent graduate with about four years of teaching experience in private tuition. In her written response to the Task she agreed with Student B's claim that the line is not a tangent. She interpreted Student A's answer as an implication of accepting the uniqueness of the common point between the line and the curve without examining the ‘nature’ of this point (she pointed out that an infinite number of lines pass through one point). She attempted to reconstruct Student A’s views through reference to graphs and then to the definition. She did not elaborate on the use of the definition; she simply cited the related formula but did not apply it in the case of the function in the Task. She accepted Student B’s graphical approach. She stressed that students are rarely at ease with the graphical approach and are often reluctant to use it. She however wrote that she would draw Student B’s attention to the fact that a graphical approach is not always feasible. Therefore, she wrote, she would demonstrate the ‘analytical’ way through an appropriate worksheet in which she would use a function with a hard-to-construct graph. For a ‘more complete repertory’ she would encourage Student A to use graphs and Student B to use the analytical approach. We invited Anna to the interview because of her emphasis on the necessity of the algebraic approach in cases where the graphical approach is not possible – not because of her concern for its validity. Also because we wanted to explore further how this sat alongside her overt appreciation of Student B’s solution.

Anna, between writing the response and being interviewed, had realised that she should accept the line as a tangent. In the interviews, she attributed her, and the students’, ‘misunderstanding of tangents’ to earlier experience with circle tangents.

> I thought that the tangent should be always like the circle tangent, but this is wrong. Because the student in question made the graph and saw it was horizontal and cuts the graph in half, he considered that this is not right, that’s why… he expected to see something like [*she gestures a line touching the graph without splitting it*].

When we asked her to describe the algebraic solution she managed only with extensive help on our part.

While discussing Student B’s response we asked Anna if she would accept Student B’s graphical solution as correct if the student had concluded with the acceptance of the line as a tangent. She said: ‘I think that we have to do all the procedure’ because ‘the line could be here, *[showing on the graph]* higher or lower, where it isn’t a tangent’ and ‘I cannot decline that it isn’t tangent but also I cannot say that it is. Don’t I have to do some…’. When we asked her why, in the light of these reservations, she accepted the graphical explanation in her *written* response, she replied: ‘I accepted it because he said that it wasn’t and I had in my mind that when I see the line splitting [the graph] there is no other choice, whatever it was’. So, would she accept a graphical solution, in general? ‘If it is correct, I would accept it’, she replied. Would she accept student B’s solution as correct if the student indicated on the graph that, although the line intersects the curve, the intersection point is an inflection point, as, for example, in the case of $f(x)=x^3$? She replied: ‘I would accept it […] it is not necessary to use the algebraic method with formulas and all that, that’s what I believe. [*hesitating*] I am not sure this is correct [*awkward laugh*]’.

She then added:

> Simply, I believe that students are not so familiar with graphical representations… and, for them, it is easier to use formulas…they see this as a methodology, as… I do not believe that they have gone into depth so that they know how to construct graphs perfectly and know how to interpret them well and this is why most of them usually use algebraic formulas. […] Because to make a graph and analyse it you have to have understood something very, very well… to own it, completely, while for this [the algebraic formula] you learn how, somewhat blindly, and you solve it, that's what I believe. In any case if [the claim] was correct I would accept it because I would see that the student understood it better than someone who can follow the algebraic formulas… now I don't know, am I right? What do you think?! [*to the interviewer*]

Later on in the interview, we asked her what would happen if the inflection point wasn't at 2.00 but very close to it (e.g. at 2.02). That made her uncertain about the accuracy of the graph. She then reconsidered her previous statement and said: 'So I believe that the best is that the students do the algebra and then make the graph [*awkward laugh*]'. She elaborated her change of mind as follows:

> I simply believe that after we solve through the algebraic formulas and find the result, then it is good to tell the students to make the graph because sometimes they reach the end and say 'ok, I found it' without having realised in their mind how it would look roughly and as soon as they see a graph they cannot answer immediately and I believe this is what happened to me… that is I was used to see circle tangents and it had crossed my mind… subconsciously that all of them must be like that … all tangents have to be like that because I was not familiar with graphs.

In the above Anna's beliefs about the acceptability or not of a visual argument appear unstable. She appears ready to accept a visual argument without any algebraic justification if the information in the image constitutes, for her, clear and convincing support for a claim. She regarded the image in the Task as sufficient evidence for determining that the line is not a tangent – also drawing on her belief that a tangent cannot intersect the graph. However she stated clearly that to prove that the line *is* a tangent an algebraic argument was necessary. Later, she stated that she could accept a *correct* statement based on the graph. When we shook her faith in the graph she declared the algebraic solution necessary. While initially she did not speak of validation of the visual statement through reference to mathematical theory, she asked for such validation when she realised that the image could be misleading.

Many times in her interview she returned to her appreciation of visual representation and argumentation as evidence of a student's in-depth understanding and as an important means towards students' construction of mathematical knowledge. She did not specify whether she meant *formal* mathematical knowledge (for example, proof). Furthermore her views with regard to the sufficiency and acceptability of a visual argument appeared rather ambivalent and heavily dependent on the specific images involved in the discussion. In this sense the didactical contract she appears to be able to offer to her students seems less clear and stable than that of Spyros.

CONCLUDING REMARKS

Spyros' clear insistence on the class' collective arrival at a formal proof as closure to the lesson is distinctly different from Anna's fluctuation between cases where she would and would not accept a visual argument. Her willingness to rely, occasionally, on imagery in order to support a claim is 'a practice that may mislead students into thinking that such are acceptable mathematical 'proofs' and reinforcing the acceptability of their empirical proof schemes.' (Harel & Sowder, 2007, p829). Furthermore, her own criteria about what makes a visual argument acceptable appeared very personal and rather fluid. Within the unstable didactical contract that this vagueness might imply, how would her students distinguish between when a visual argument is acceptable and when not? In the already compounded didactical contract of school mathematics such vagueness can be detrimental.

A clearer contract could be as follows: in a classroom discussion where a visually-based (incorrect) claim is proposed, the class employs the algebraic, formal approach to convince the proposer about the incorrectness of their claim. Even when a visually-based (correct) claim is unequivocally accepted by the whole class, the class still employs the algebraic approach to establish the validity of the claim formally. In both cases visualisation emerges as a path to insight and proof as the way to collectively establish the validity of insight. In both cases there is a pedagogical opportunity for linking imagery with algebra and for embedding the algebra in the immediately graspable meaning in the image.

The above suggest a role for proof in the mathematics classroom that is not disjoint from the creative parts of visually-based classroom activity and that reflects an essential intellectual need. We conclude with quoting Harel & Sowder's (2007, p836) statement regarding this intellectual need:

> The subjective notion of proof schemes is not in conflict with our insistence on unambiguous goals in the teaching of proof – namely, to gradually help students develop an understanding of proof that is consistent with that shared and practised by the mathematicians of today. The question of critical importance is: What instructional interventions can bring students to see an intellectual need to refine and alter their current proof schemes into deductive proof schemes.

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THE ALGEBRAIC MANIPULATOR OF ALNUSET: A TOOL TO PROVE

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This report is devoted to analyzing the influence of an algebraic system, the Algebraic Manipulator of ALNUSET on students' construction of proof in proving equivalence among expressions. Results of an experiment, carried out with students at the second year of Upper Secondary school, are presented to show in which way this manipulator can be used in the educational practice to enhance the teaching and learning of algebraic proof.

INTRODUCTION

As underlined in the introduction to the special issue of ZDM on didactical and epistemological perspectives on mathematical proof (Mariotti and Balacheff, 2008), research work about mathematical proof has been growing in the last decade. Different perspectives (historical and epistemological issues, cognitive ones, didactical transposition of mathematical proof into the classroom) are taken into account framing the proof from different points of view. The actual invitation addressed to educational researchers is to find complementarities in this variety of approaches to make them converge (Balacheff, 2008). This required effort has double goal. On one hand, it could mean an acknowledged awareness of what connects and what separates our works, and on the other hand, it could strongly contribute to teaching and learning of proof in everyday classes. Finally, this effort could make possible the connection between educational research and the school context making our research work effective and fruitful.

Due to my concern for this aspect, I have been studying to find "effective supports" to the didactical transposition of mathematical proof into the classroom. Starting from evidence highlighted by existing research works about students' difficulties in approaching proof, I show a possible way to use technological artefact in the classroom to support the teaching and learning of proof effectively. In this report I present a part of this work in progress. In particular, some interesting results of an experiment carried out with students at the second year of Upper Secondary School are reported.

STUDENTS DIFFICULTIES IN LEARNING PROOF

Students' difficulties in learning mathematical proof have been pointed out by many different research works. In this report I am particularly interested in two of them: students do not see the usefulness of a mathematical proof and they do not understand its language and symbolism.

A new balance between the need to produce logical argument and the need to provide an argument that explains, communicates and convinces seems to be necessary (Healy and Hoyles, 2000). Various authors point out the importance of the explicative and justificative roles of proof (Hanna, 1989, 2000, Harel and Sowder, 1998) that often are not grasped by students. The importance of proof should go beyond the establishment of mathematical truth. A broader vision of proof is expected: proof should provide students with important mathematical strategies and methods for solving problems. (Hanna and Barbeau, 2008).

This new approach to proof could effectively support students in seeing the usefulness of a mathematical proof but other difficulties could come out and they have to be considered. For example, the deductive nature of proof and its symbolism should be explained and justified too. Research results highlighted the great difference between argumentation and proof both from a semantic point of view (Duval, 1995) and from a structural one (Pedemonte, 2007); it is important to distinguish between truth and validity from a logical point of view (Durand-Guerrier, 2008). Logical structure, language and symbolism are important aspects in the construction of proof but they remain often hidden for students. Proof can appear to students as a sub-minimal code with no vital information for understanding (Alibert and Thomas, 1991).

Furthermore, some studies highlight the role of the proof as theoretical organization. These studies focus on the importance of introducing students to the axiomatic structure of proof and to a theoretical perspective (Mariotti & al., 1997). Their aim is to help students access the meaning of theorem and support them in the transition from the need of justifying to the need of validating within a mathematical system (Mariotti & al., 1997).

In general, all these studies show that the role of proof in the educational practice is not well defined and very often difficulties emerge because some aspects of proof are not explicit for students and they are not well explained by teachers.

In teaching proof, certain often implicit aspects need to become part of explicit educational goals (Hemmi, 2008). Through the notion of "transparence", Hemmi contributes to solve the dilemma to make more or less visible to students some important aspects concerning proof. The concept of transparency (Lave and Wenger, 1991) combines two characteristics: visibility and invisibility. Visibility concerns the ways that focus on the significance of proof (construction of the proof, logical structure of proof, its function, etc.). Invisibility is the form of "unproblematic interpretation" and integration to the activity (Hemmi, 2008, p. 414). It concerns the proof as a justification of the solution of a problem without considering it as a proof. It has been underlined that "*Proof as an artifact needs to be both seen (to be visible) and used and seen through (to be invisible) in order to provide access to mathematical learning*" (Hemmi, 2008, p. 425). The lack of transparency in the teaching of proof regards the lack of knowledge about proof techniques, key ideas and proof strategies.

These considerations offer important insights to make the transposition of mathematical proof into the classroom effective.

In this context I intend to contribute through the Algebraic Manipulator of Alnuset. This system can be used in teaching and learning algebraic proofs to make rules and axioms used visible in proof processes and to make theoretical aspects usually implicit in algebraic manipulation emerge. The aim of this report is to show in which way the Algebraic Manipulator can be used in the educational practice to enhance the teaching and learning of algebraic proof.

ALNUSET

Alnuset is a system developed in the context of ReMath (IST - 4 - 26751) EC project for students of lower and upper secondary school (yrs 12-13 to 16/17). It is constituted by three integrated components: the Algebraic Line component, the Algebraic Manipulator component, and the Functions component. Even if the educational relevance of this system emerges better through the integrated use of these three components, in this paper I only consider the Algebraic manipulator component to show how it can be used to modify the approach to the algebraic proof.

To have a more complete idea about this system you can see the report presented in group 7 by Chiappini G., and Pedemonte B. of this edition of CERME.

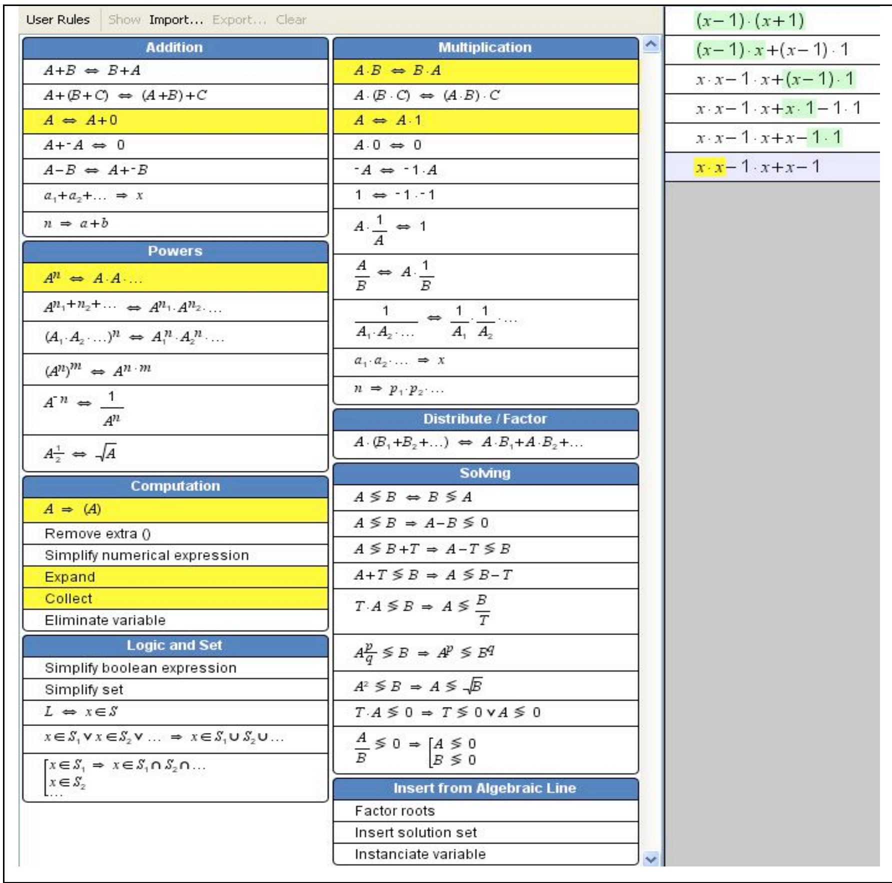
The Algebraic Manipulator of Alnuset: a tool to prove

The Algebraic Manipulator component (AM) of Alnuset is a structured symbolic calculation environment for the manipulation of algebraic expressions and for the solution of equations and inequations.

Its operative features are based on pattern matching and rewriting rules techniques. In the AM these techniques are used in a different perspective with respect to the CAS where the basic rules (commutativity, associativity, etc.) are used internally in a sequence generally not controlled by the user, to produce a higher level result, like “factorize” or “combine”. As a consequence, the techniques of transformation involved in CAS can be obscure for a non expert user.

In the AM, pattern matching is based on a structured set of basic rules that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. These rules are explicit for students. They appear as commands on the interface made active only if they can be applied to the part of expression currently selected.

An expression is transformed into another through this set of rules. Students can see the transformation of an expression as result of the application of a rule to it.



A sequence of rules (chosen from the left panel) are applied to the initial expression (x-1)(x+1). At each step, the rule is applied to the green sub-expression, producing the expression on the next line. The last line shows the current selection (x*x in yellow), and one of the 7 rules highlighted in yellow can be applied to this sub-expression.

Moreover, the system allows the student to create new transformational rules (user rules) once these new rules have been previously derived. This feature also present in the L'Algebrista (Cerulli, Mariotti, 2003) is important because it can be used to construct an idea of structured theory.

In the following I show how the AM can be used to provide a good "transparency" (Hemmi, 2008) for the concept of proof. This system can be used to introduce proof in Algebra making visible the rules and procedures of manipulation supporting the comprehension of proof as part of a theoretical system. Moreover, the AM could be used to propose problems involving proof without a direct focus on it. For space reasons, in this report only the role of Alnuset as tool allowing the "visibility" of some important concepts about algebraic proof is analysed.

TEACHING EXPERIMENT

In this section, students' resolution processes of some tasks involving the construction of proof in the AM of Alnuset are analysed. They are taken from a set of data collected from an experiment carried out in a class of 24 students of the second year of Upper Secondary School (15-16 years old) in the context of ReMath EC project.

The main aim of this experiment was to analyse the role of Alnuset in a teaching experiment centred on algebraic expressions and propositions. The experiment lasted ten weeks, with a 2-hour section each week. The first part of the teaching experiment focused on algebraic expressions (equivalent expressions, opposite expressions, reciprocal expressions). In this part, a specific section was devoted to the manipulation of expressions. In this report I present results of this section.

During the previous weeks students had used the AM of Alnuset only twice.

Students worked in pairs with the AM of Alnuset under the supervision of the teacher and the researcher.

In the following, tasks proposed to students during the section are presented.

Tasks

a) Use AM to prove that (2+3)*5-25 is equal to 0
Use AM to prove the same equality starting by 0.
Is this the only equivalence that it is possible to prove starting by 0?

b) Use AM to prove that (2/5+4/5)*5/6 is equal to 1
Use AM to prove the same equality starting by 1.
Is this the only equivalence that it is possible to prove starting by 1?

c) In solving tasks a) and b) you have used two specific commands, both in direct and indirect ways: the command to add two opposite expressions (A+-A⇔0) and the command to multiply two reciprocal expressions (A*1/A⇔1). Have you observed any difference in the direct and indirect use of these commands? If yes, what differences? In your opinion, is it more difficult to accomplish proofs based on the direct use or proofs based on the indirect use of these commands? Why?

d) Try to prove that the expression a/b+c/d is equivalent to the expression (a*d +b*c)/bd. If this proof is difficult for you, try to prove the equivalence between the two expressions starting from (a*d +b*c)/bd and then to come back step by step in order to work out the more complex proof.
Use the accomplished proof to create a new manipulation rule.

e) Try to prove that the expression a^2-b^2 is equivalent to the expression (a+b)(a-b). If this proof is difficult for you, try to prove the equivalence between the two expressions starting from (a+b)(a-b) and then to go backward, step by step, in order to work out the more complex proof. Use the accomplished proof to create a new manipulation rule.

f) Use AM to transform the following expressions using, if necessary, the rules created in the previous tasks:
x^2-4; x^2-1; $\frac{x+2}{x+1}+\frac{x+1}{x-2}$

Tasks a) and b) introduce the two rules A+-A⇔0 and A*1/A⇔1 instantiated on specific examples. Task c) supports reflections about the direct and indirect use of these rules. Tasks d) and e) require to prove the rules a/b+c/d = (a*d +b*c)/bd and a^2-b^2 = (a+b)(a-b) using the two rules A+-A⇔0 and A*1/A⇔1. Task f) is useful to strengthen the use of the new proved rules.

<u>Tasks a), b) and c)</u>

The solution of task a) in the manipulator is reported in the following table. In the first part there is the manipulation from the numerical expression to 0 and in the second part there is the manipulation from 0 to the expression.

$(2+3)\cdot 5-25$	0	The second proof (right) is more difficult for students with respect to the first one (left). In the second proof, the equivalence needs a step that obliges the user to write 0 as addition of two opposite numbers (25-25). This is not obvious for students who in general are not able to manage it.
$(5)\cdot 5-25$	$25+-25$	
$25-25$	$25-25$	
$25+-25$	$5\cdot 5-25$	
0	$(2+3)\cdot 5-25$	

The application of the rule 0=>A+-A requires to understand that 0 can be expressed as sum of two opposite expressions. The problem is that there are infinite possibilities that can be considered to replace 0.

In the same way, to apply the rule 1=>A*1/A students have to replace 1 with two reciprocal expressions.

$(\frac{2}{5}+\frac{4}{5})\cdot\frac{5}{6}$	1	The second proof (right) is more difficult for students with respect to the first one (left). In the second proof, the equivalence needs 2 steps that oblige the user to write 1 as multiplication of two reciprocal numbers (5*1/5 and 6*1/6). As in the previous case, this is not obvious for students who are not able to replace 1.
$(2\cdot\frac{1}{5}+\frac{4}{5})\cdot\frac{5}{6}$	$6\cdot\frac{1}{6}$	
$(2\cdot\frac{1}{5}+4\cdot\frac{1}{5})\cdot\frac{5}{6}$	$6\cdot 1\cdot\frac{1}{6}$	
$((2+4)\cdot\frac{1}{5})\cdot\frac{5}{6}$	$6\cdot 5\cdot\frac{1}{5}\cdot\frac{1}{6}$	
$((6)\cdot\frac{1}{5})\cdot\frac{5}{6}$	$6\cdot\frac{1}{5}\cdot 5\cdot\frac{1}{6}$	
$6\cdot\frac{1}{5}\cdot\frac{5}{6}$	$6\cdot\frac{1}{5}\cdot\frac{5}{6}$	
$6\cdot\frac{1}{5}\cdot 5\cdot\frac{1}{6}$	$(2+4)\cdot\frac{1}{5}\cdot\frac{5}{6}$	
$6\cdot 1\cdot\frac{1}{6}$	$(2\cdot\frac{1}{5}+4\cdot\frac{1}{5})\cdot\frac{5}{6}$	
$6\cdot\frac{1}{6}$	$(\frac{2}{5}+4\cdot\frac{1}{5})\cdot\frac{5}{6}$	
1	$(\frac{2}{5}+\frac{4}{5})\cdot\frac{5}{6}$	

As shown by the results of the experiment, in general these rules are used by students in their manipulations in paper and pen environment, in a completely implicit way. Most students are able to transform an expression into another one using these rules but they are not able to explicit them. In better cases they are able to use these rules as computational techniques but they are rarely able to justify them.

Analysis of results of tasks a), b) and c)

The results analysis of the experiment shows that most students constructed the direct proof in tasks a) and b) even if for task b) the intervention of the teacher was often necessary. Students knew the result of the sum 2/5+4/5 but they were not able to make it in the AM because they didn't manage the properties and rules hidden in the technique of addition of two fractions.

The construction of the inverse proofs (from 0 to the expression (2+3)*5-25 and from 1 to (2/5+4/5)*5/6) was not easy for them. As expected, difficulties emerged when students had to replace 0 as sum of two expressions and 1 as multiplication of two expressions.

Only observing the previously constructed direct proof some students (6 groups out of 12) were able to construct also the inverse proof, following step by step the direct proof and going backwards to the initial expression. Here is the dialog between two students while constructing the proof from 0 to the expression (2+3)*5-25.

> I: But in which way can we prove this equivalence starting from 0?
> F: perhaps…
> I: wait a moment… if a+-a is 0 it is also true that 0 is a+-a
> F: yes, of course
> I: then if 25-25 is 0 it is also true that 0 is equal to 25-25... then we can write in this way
> F: following step by step the previous proof

The AM allowed students to make explicit rules A+-A⇔0, A*1/A⇔1 and to understand the intrinsic difference that characterises the two directions of the rules. Let's see the following example (answers reported in the copy of a group of student):

> "a) Starting with 0 it is possible to prove whatever equivalence having 0 as result. So there are infinite equivalent expressions to 0. b) Starting with 1 it is possible to prove that 1 can be replaced by all reciprocal expressions having 1 as results. c) In our opinion it is easier to produce proofs based on the direct use of the command A+-A ⇔ 0, because in the inverse case it is necessary to look for the opposite expression, while the direct use of the command only requires the application of the correct axiom. For the rule A*1/A ⇔ 1 the principle is the same, but in this case consider reciprocal expressions and not opposite expressions".

Answers given by these students to task c) show that they have developed awareness about the role of the two rules and the way they can be used in manipulation.

Tasks a), b), and c) allowed students to reflect deeply on these rules that are usually used in the algebraic manipulation in a completely "invisible" way. The AM of Alnuset allowed students to "make visible" these rules and their use in the construction of the proofs.

<u>Tasks d), e) and f)</u>

Task d) and task e) are very useful in approaching proof and in particular they are effective to understand the idea of theoretical systems. As a matter of fact, only when the rules a^2-b^2 = (a+b)(a-b) and a/b+c/d =(a*d +b*c)/bd are proved they can become new user rules and they can be used to prove expressions as those proposed in task f).

A possible solution of the task e) in the AM is reported in the following table.

a^2-b^2	$(a+b)\cdot(a-b)$	It is better to begin from the second proof (right) because in the first proof (left) it is necessary to insert 0 and replace it with the sum of the two opposite expressions ab and –ab.
a^2+0-b^2	$(a+b)\cdot a-(a+b)\cdot b$	
$a^2+a\cdot b+-(a\cdot b)-b^2$	$a\cdot a+b\cdot a-(a+b)\cdot b$	
$a\cdot a+a\cdot b+-(a\cdot b)-b^2$	$a\cdot a+b\cdot a-(a\cdot b+b\cdot b)$	
$a\cdot a+a\cdot b+-(a\cdot b)-b\cdot b$	$a\cdot a+b\cdot a+-(a\cdot b+b\cdot b)$	
$a\cdot(a+b)+-(a\cdot b)-b\cdot b$	$a\cdot a+b\cdot a+-1\cdot(a\cdot b+b\cdot b)$	Once the proof is accomplished students can solve it as a new rule: the following one.
$a\cdot(a+b)+-1\cdot(a\cdot b)-b\cdot b$	$a\cdot a+b\cdot a+-1\cdot a\cdot b+-1\cdot b\cdot b$	**User Rules**
$a\cdot(a+b)+-1\cdot a\cdot b-b\cdot b$	$a\cdot a+b\cdot a+-(a\cdot b)+-1\cdot b\cdot b$	$a^2-b^2 \Leftrightarrow (a+b)\cdot(a-b)$
$a\cdot(a+b)+-1\cdot a\cdot b+-(b\cdot b)$	$a\cdot a+a\cdot b+-(a\cdot b)+-1\cdot b\cdot b$	
$a\cdot(a+b)+-1\cdot a\cdot b+-1\cdot(b\cdot b)$	$a\cdot a+0+-1\cdot b\cdot b$	This user rule can be used in the successive manipulations.
$a\cdot(a+b)+-1\cdot a\cdot b+-1\cdot b\cdot b$	$a\cdot a+-1\cdot b\cdot b$	
$a\cdot(a+b)+-1\cdot b\cdot a+-1\cdot b\cdot b$	$a^2+-1\cdot b\cdot b$	
$a\cdot(a+b)+-1\cdot(b\cdot a+b\cdot b)$	$a^2+-1\cdot b^2$	
$a\cdot(a+b)+-1\cdot(b\cdot(a+b))$	$a^2-1\cdot b^2$	
$a\cdot(a+b)+-1\cdot b\cdot(a+b)$	a^2-b^2	
$(a+-1\cdot b)\cdot(a+b)$		
$(a+-b)\cdot(a+b)$		
$(a-b)\cdot(a+b)$		
$(a+b)\cdot(a-b)$		

A lot of steps are necessary to prove the equivalence a^2-b^2 = (a+b)(a-b) in the AM of Alnuset, because manipulation requests students to make rules and axioms that are necessary to prove the equivalence explicit.

In the same way it is possible to produce the proof of the equivalence a/b+c/d = (a*d +b*c)/bd.

<u>Analysis of results of tasks d), e) and f)</u>

Tasks d) and e) required a lot of efforts by students. Nevertheless, these tasks were very fruitful to understand the meaning of proving a rule starting by a basic set of rules and axioms. Students who tried to prove the two equivalences a/b+c/d=(a*d +b*c)/bd and a^2-b^2=(a+b)(a-b) inserting the first expression (a/b+c/d or a^2-b^2) were not able to begin the manipulation. All students were forced to follow the suggestion given by the text of the tasks inserting the second expression and manipulating it. Also in this case the solution was not obvious. Some difficulties concerned denotative aspects: deletion of superfluous parentheses, application of properties in order to make the expression match with the rule to be applied, and so on. Nevertheless, in some cases, difficulties concerned "conceptual aspects" usually invisible in the ordinary manipulation in the paper and pen environment. For example, students were not confident with rules such as a-b=a+-b and -a=-1*a. Thus steps concerning the application of these rules were often introduced by the teacher. Let's see the dialogue of two students during the resolution of task d).

S: This is a specific product… Insert in Alnuset the expression a^2-b^2
L inserts the expression in AM
S: and then?
L: I really have no idea….
S tries to apply some rules without success.
L: Perhaps… it is better to start from the other side. Try to insert (a-b)(a+b)
S inserts the expression in AM and then she applies the distributive law.
She is not able to sum –ab +ab because she was not able to transform the expression aa+ba-(ab+bb) into the expression aa+ba-ba-bb.
L: What? We are not able to add these two expressions. We know that the solution is 0 but...
S: in which way can we find this result?
Teacher: You have to apply the rules a-b=a+-b to transform –(ab+bb) into -1(ab+bb)…
S: Ah ok! We try…
Students complete the proof and they try to perform the inverse proof.

Even if it was really hard for students to solve the tasks, the constructed proofs obliged them to make explicit axioms and rules that are used step by step during the transformation of an expression into another.

In general, students were very proud of their proofs and they liked a lot to save the proved rules as new rules that could be used in their successive proofs. Task f) was solved by most students without any difficulty. In this task they eventually realised that the previously proved rules were useful to prove other new rules.

CONCLUSIONS

The results of the experiment might show that the AM of Alnuset does not help students construct proofs and makes proofs more complicate for them. In a sense this is true - a lot of students are able to transform (a+b)(a-b) into a^2-b^2 in paper and pen environment and perhaps it is not so important to be able to make the inverse transformation. The problem is that in school practice, algebra is usually considered as a body of rules and procedures for manipulating symbols. Students are usually able to develop calculus but they are not aware of the axioms and theorems they are using in performing it. Thus, algebra is taught and learned as a language and emphasis is put on its syntactical aspects. In this context, algebraic proof appears as a grammar structure made of a sequence of formulae connected by calculus rules. In this way, the meaning of proof is completely lost. Despite this, rigorous proof is generally considered as a sequence of formulae within a given system, each formula being either an axiom or derivable from an earlier formula by a rule of the system. The AM of Alnuset supports this kind of proof though in a different way. Each step in the manipulation is produced by the application of a rule that has to be chosen by the student from a set of rules. If the choice is not correct it could be very difficult for the student to construct the proof. During the experiment the intervention of the teacher often supported students that were unable to accomplish the task. Notwithstanding this, at the end of the experiment, students were able to explicit rules used during their proofs spontaneously. Also during ordinary school practice, students justified their steps making the rule used in the transformation explicit. This kind of approach required a lot of effort but it supports the awareness of what it is an algebraic proof and in which way a mathematical theory can be constructed.

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MATHEMATICS TEACHERS' REASONING FOR REFUTING STUDENTS' INVALID CLAIMS

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This study investigates secondary school mathematics teachers' reasoning for refuting students' invalid claims in the context of hypothetical classroom scenarios. The data used in this paper comes from seventy six teachers' written responses to a (hypothetical) student's invalid claim about congruency of two given triangles and from interviews with a number of them. Analysis of the responses led to the identification o two main approaches to refuting the student's claim: by using known theorems or by providing counterexamples. The participating teachers encountered difficulty in generating counterexamples. Moreover, the teachers employed a rather narrow meaning of the theorems and their use to refute invalid claims.

INTRODUCTION

Reasoning and proof are considered fundamental aspects of mathematical practice both in the practice of mathematicians and in the practice of students and teachers (Hanna, 2000). A large number of studies in mathematics education explored students' justifications and proof strategies (eg. Healy and Hoyles, 2000; Harel and Sowder, 1998). Refuting conjectures and justifying invalid claims requires reasoning that goes beyond the syntactic derivations of deductive proof which has been traditionally the focus of high school mathematics. It mainly involves the generation of counterexamples, and the development of logical arguments that are grounded in exploration and experimentation, which are related to the construction of mathematical meaning and understanding. Balacheff (1991) discusses the diversity of ways of dealing with a refutation by referring to Lakatos's (1976) epistemological work and to his own experimental study with high school students. Lin (2005) also demonstrates the complexity of the process by identifying the different types of arguments that secondary school pupils developed to refute false conjectures.

The process of evaluating and refuting students' claims is central to teacher practice. This often requires the teacher to give on the spot appropriate explanations that involve the use of examples or counterexamples. The process of exemplification is highly demanding but it has not been extensively investigated with regard to the teacher (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006). Desirable choice of examples depends on teacher's subject matter knowledge (Rowland, Thwaites & Huckstep, 2003) on her teaching experience (Peled & Zaslavsky, 1997) and on her awareness of students' prior experience (Tsamir & Dreyfus, 2002). The generation of examples and counterexamples in geometry gets a special meaning as the visual entailments of examples pose certain constraints (Zodik & Zaslavsky, 2008). In this paper, we investigate how teachers respond to students' invalid claims in the context of Euclidean geometry.

THEORETICAL BACKGROUND

We mention below the main theoretical constructs that framed our study. These concern teacher knowledge, the process of refutation and the nature and the use of counterexamples.

The process of evaluating and refuting students' invalid claims relates to mathematics teacher knowledge. Stylianidis and Ball (2008) studied the characteristics of teacher knowledge for reasoning and proof. Zodik and Zaslavsky (2008) also attempted to capture the dynamics and complexity involved in secondary mathematics teachers' choice and generation of examples in the course of their teaching.

The process of refutation has been mainly studied under the epistemological framework of Lakatos (1976) (eg. Balacheff, 1991; Larsen & Zandieh, 2007). Lin (2005) developed a categorisation of students' refutation schemes. He distinguished rhetorical arguments (reasons relative to the person spoken), heuristic arguments (correcting the given information), and mathematical proofs (the process of generating counterexamples).

Peled & Zaslavsky (1997) distinguished three types of counterexamples suggested by mathematics teachers: specific, semi-general and general examples. Semi-general and general examples offer some explanation and ideas to generate more counterexamples. Related to teacher's generation of counterexamples is the theory of personal example spaces which is what is accessible in response to a particular situation (Bill et al, 2006). Zazkis & Chernoff (2008) introduced the notions of pivotal examples and bridging examples and highlighted their role in creating and resolving a cognitive conflict.

The study reported here is part of a larger study that investigates teachers' ways of refuting students' false conjectures. In this paper, we explore the different types of arguments that teachers use in refuting an invalid claim in the context of geometry, an area where research is rather limited.

METHODOLOGY

Seventy six teachers who were all candidates for a Masters in Mathematics Education programme participated in the study. Six of them were primary school teachers with an education degree, while the rest were secondary school teachers with a mathematics degree. Some were novice teachers while most had more than ten years of teaching experience.

The teachers took a three hour exam as part of the selection process for the Masters programme. In this exam they had to deal with five written tasks where they were asked to react to hypothetical teaching events. Four of these hypothetical events were related to the process of refuting students' conjectures. Then we analysed each teacher's written responses in terms of mathematical and pedagogical issues. On the base of this analysis, 45 teachers were interviewed individually in order to explore further their choices and justifications. Each interview lasted about 15-30 minutes.

One of the researchers interviewed the teachers when one of the other two was taking field notes. In order to avoid negative effects on the candidates the interviews, which constituted part of the selection process, were not audio or video recorded.

Our analysis of the above interviews yielded 5 categories of teachers' responses to the student's invalid claim. In order to investigate more deeply teachers' reasoning we organised more extended interviews with two teachers from each of the five categories that were formed. The last interviews were not part of the selection process, so they were audio-taped and transcribed.

In this paper we analyse the data based on the test and the first interviews concerning one of the tasks.

The task

The task was the following:

> In a Geometry lesson, in grade 10, the teacher gave the following task:
> Two triangles ABC and EFG have BC=FG=12 and AB=EF=7 and the angles ACB and EGF equal to 30 degrees. Examine if the two triangles are congruent.
> Two students discussed the above task and expressed the following opinions:
>
> Student A: The two triangles have two sides and an angle equal. Therefore they are congruent.
>
> Student B: We know from the theory that two triangles are congruent when they have two sides and a contained angle equal. Therefore, the given triangles are not congruent.
>
> If the above dialogue took place in your classroom, how would you react?

The task refers to a hypothetical classroom scenario which focuses on issues of learning and teaching mathematics. Further discussion about the importance of this type of tasks as a research tool for exploring teachers' thinking can be seen in Biza, Nardi & Zachariades (2007). This task was based on an example discussed and analysed by Zodik & Zaslavsky (2007). Its mathematical content, the properties of the triangles and their congruence, is part of the Euclidian Geometry course taught in grade 10 in Greek high schools. In the task, student A believes that if one triangle has two sides and an angle equal to two sides and angle of another triangle then the two triangles are congruent. He seems to over generalize the theorem "if two triangles have two sides and the contained angle equal then they are congruent." There are at least three different approaches to refute the claim of this student. The first one is to provide a specific counterexample based on a geometric construction using a ruler and compass (Figure 1).

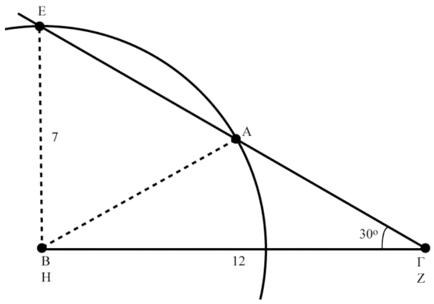

Figure 1: A geometric construction of a counterexample

In this case we may continue and prove a general geometric theorem based on the geometrical construction - there are exactly two distinct triangles that are equal in two of their sides and the angle opposite the shorter of the two. If the angle is opposite the larger side or if the sides are equal then the triangle is uniquely determined, that is, any other one satisfying these conditions is congruent to it.

The second approach is to prove this general theorem and apply it to the specific given case. The third approach is to use the sine and cosine law in trigonometry. By applying the cosine rule for the given angle, we determine the third side, and find that there are two possible values for its length. By applying the sine law we find that there are two possible angles – an acute one and its complementary. An interpretation of this calculation and the verification of the existence of triangles with these sides or angles lead to the conclusion that there are two (and only two) distinct non-congruent triangles satisfying the givens.

RESULTS

Initial analysis of teachers' written responses

In this section, we classify teachers' justifications based on their written responses. Out of the seventy six mathematics teachers three did not reply while eight considered the given triangles congruent. The remaining sixty five teachers acknowledged that the given triangles were not necessarily congruent. Sixty-three of them gave an explicit justification to their claim. These justifications were grouped in categories which are demonstrated in the tree diagram of Figure 2. The numbers in brackets show the number of responses that fall in each category.

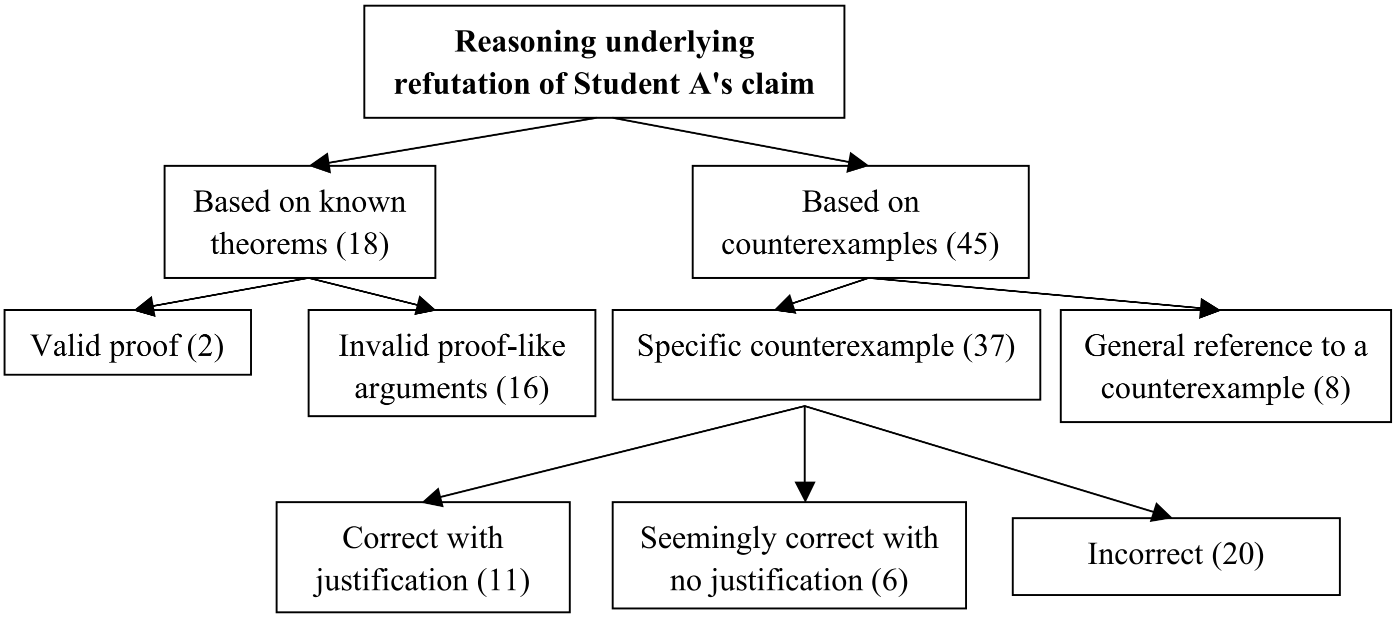

Figure 2: Mathematics teachers' justifications

Out of 63 teachers, 18 justified their claim by drawing on geometrical theorems relevant to the problem and 45 asserted that a counterexample was needed to justify their claim. A Binomial Test indicates significant differences between these two approaches (p=0.001). This result reflects teachers' tendency to use counterexamples in order to refute claims. However, only a small number of teachers (11 out of 45) constructed an appropriate counterexample and justified their construction, a difference which is also statistically significant (p = 0.001).

Reasoning based on known theorems

Valid proof: Out of the 18 teachers who fall in this group two gave a valid proof. One of them (T64) used the sine rule and proved that one pair of corresponding angles of the two triangles are equal or complementary without exploring further if both cases are possible. Another one (T32) used the cosine rule to calculate the length of the third side and found two possible values, thus concluded that two different triangles exist with the same givens.

Invalid proof-like arguments: The remaining sixteen teachers provided invalid proof-like arguments to support their claim by maintaining that none of the known theorems about the congruence of two triangles applies in this case. The following example indicates the latter case: "Student A replied without considering the known criteria for congruence of triangles. I would encourage him to draw the two triangles so that to realise that these criteria cannot be applied" (T10). These teachers believed that this reasoning offers a valid proof for refuting student A's claim.

Reasoning based on counterexamples

General reference to a counterexample: Out of the 45 teachers who fall in this group, 8 made only a reference to the need to give a counterexample by stating that they themselves or their students would give a counterexample. For example, T23 simply mentioned that "... to convince him (Student A) we could show him some triangles that have two sides and one angle equal but are not congruent" while T18 suggested asking the students: "... to experiment with the shapes and to make many different trials. So, Student A will see a good counterexample that will contradict his view".

Specific counterexamples: The remaining 37 teachers, half of the participants, gave a specific counterexample. Twenty of these teachers provided *incorrect counterexamples*. Some sketched two triangles that supposedly satisfied the given conditions and claimed that these triangles were not congruent. However, these triangles seemed congruent – thus were non-appropriate or had too many constraints - thus were non-existent. For example, T72 drew two triangles that seemed symmetrical in his attempt to produce two triangles that were not congruent (Figure 3).

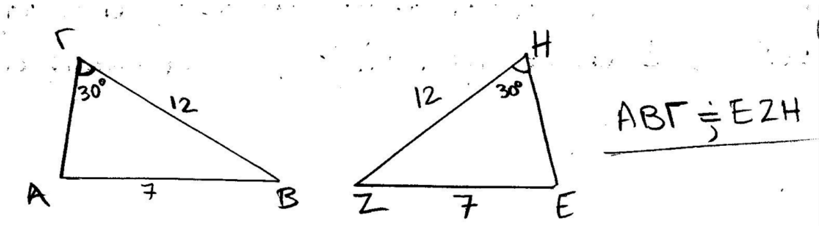

Figure 3: The drawing of T72

Some teachers considered the variation of a pair of angles but without giving specific measures as in the case of T39. This teacher sketched two triangles and she commented: "If we draw on the board two triangles ABC and EZH with the given elements and the angles ABC and EZH to be acute - one smaller than the other – it is easy to verify by using transparent paper that the two triangles are not congruent". Others drew two triangles by attributing specific values to the angle contained between the two given sides. For example, T74 wrote: "I would ask the students to make two triangles ABC and EHZ with BC=ZH=12, AB=EZ=7, the angles ACB and EHZ equal to 30 degrees, the angle ABC equals 90 degrees and the angle EZH equals 45 degrees". In both cases, as it appeared also from the interviews and will be analysed further below, the teachers did not think about the existence of the suggested triangles.

Six teachers gave counterexamples *seemingly correct with no justification*. They drew two triangles which satisfied the given conditions with the angles opposite to the sides of 12 cm to seem complementary, like in the appropriate counterexample. They claimed that this was a counterexample but did not give any justification for their claim. Finally, eleven teachers gave a *correct counterexample with a valid justification* by constructing geometrically the two triangles that had the given elements and were not congruent. Some of them suggested to explore with the students further the situation and to construct relevant theorems. For example, after the geometrical construction of a counterexample T33 wrote: "I would ask the students to try to prove that if one triangle has two sides and the angle opposite to one of these sides equal to the corresponding sides and angle of another triangle, then the corresponding angles that are not contained in the two sides are either equal or complementary".

Emerging epistemological issues

The (non-)existence of teachers' generated counterexamples

In the interviews, the teachers who had not justified the process of constructing a counterexample as well as those who gave an incorrect example, were asked about its existence: "How do you know that the triangles you have drawn exist?". Some of them argued for the existence of their counterexample by recalling a theorem or an image or by describing the drawing process. Examples of their arguments are the following:

"I have seen this counterexample in a textbook" (T32, recalling an image)

"If I remember well, there is a theorem that says that the non-contained angles are equal or complementary". (T40, recalling a theorem)

"The sum of their angles is 180 degrees...They can be constructed...I can vary the angles" (T69, recalling a theorem)

"I made them; I measured by the ruler its sides". (T30, describing the drawing process)

Many teachers had not considered the problem of the existence in their initial responses. In the interview, most of them responded immediately that they had to check the existence of the suggested triangles. However, there were some who seemed to believe that the question about the existence of a triangle with specific properties had no meaning. Typical responses were:

"Yes, why can't we? Do we have to prove it?" (T46)

"Is it possible not to exist?" (T60)

"I thought that it is sure that there are two triangles [satisfying the given conditions] which are not congruent. So, I opened a bit the angle and I moved the side to that direction." (T73)

Another issue that emerged and was related to the problem of existence was the number of possible counterexamples. There were teachers who believed that there was more than one counterexample and in some cases they described a process of generating an infinite number of triangles (for example, T69, T73 mentioned above). This finding concurs with the findings of Zodik & Zaslavsky (2007).

During the interviews, we observed that some of these teachers started to think about ways of constructing appropriate counterexamples. For example, T39 had initially sketched two acute-angled triangles and she had written that "two triangles with acute angles, one smaller than the other are not congruent as it can be verified by a transparent sheet". In the interview she formed a new hypothesis that: "if both triangles [satisfying the given conditions] have all their angles acute, they are congruent while they must be different if one triangle is acute-angled and the other obtuse." Later in the interview, she used the sine-rule trying to prove her hypothesis. However, she did not manage to construct geometrically the suggested triangles. On the other hand, T32 had given as a counterexample two triangles, one right-angled and the other isosceles. In the interview she immediately recalled a known theorem "that one pair of angles can be equal or complementary" and finally she gave a correct geometrical construction of the counterexample.

Teachers' use of theorems to refute a claim

Another issue that emerged from our data was the use of theorems for justifying refutable (invalid) claims. A number of teachers believed that the non applicability of the known relevant theorems implied that the claim was wrong. In particular, some teachers concluded that the two triangles were not necessarily congruent as none of the three commonly used criteria about the congruence of triangles (S-A-S, A-S-A, & S-S-S) could be applied. For example, in his written response T11 reminded the students these three criteria, and added that "the problem statement does not satisfy the criterion S-A-S ... so from the given data we cannot conclude that the two triangles are congruent". The above argument was the only one that the teacher gave for justification. T47 also expressed a similar view in his written response. In the interview, although this belief was challenged by one of the researchers, it seemed to be rather strong as demonstrated in the following extract:

> T47: The two triangles are not necessarily congruent.
> R: How do you know this?
> T47: We cannot apply the criterion S-A-S.
> R: Ok, a known criterion cannot be applied. But how do you know that there is no other way to prove the congruence of the two triangles?
> T47: We cannot prove the congruence with the criteria we teach.

Here, the teachers seem to base their reasoning on the principle that we can prove that two triangles with given properties are congruent only if these properties satisfy one of the three above commonly used criteria. So, in order to prove that the two triangles are not necessarily congruent, they assumed the negation of this claim and they came to a contradiction to the above principle because none of the three criteria could be applied. However, the above reasoning is not a correct mathematically and can lead us to wrong conclusions. For example, if the length of one of the sides of the two triangles of our problem were 6 instead of 7, then the above reasoning would lead to the conclusion that the two triangles are not congruent. However, in this case the two triangles are right-angled and thus are congruent. The above way for refuting invalid claims is not a mathematical proof but in some cases can be used as a tool for making conjectures. For example, in his written response T21 initially stated that the known criteria could not be applied and then gave a geometrical construction of the counterexample.

Another use of theorems was for developing a deductive proof for supporting the claim that the two given triangles were not necessarily congruent. T64 used the sine rule in the given triangles and came to the conclusion that the angles opposite the side of length 12 are equal or complementary. T32 used the cosine rule to calculate the third side and found two possible lengths. Clearly, these proofs need to be followed by the verification of the existence of the two possible triangles.

These two approaches are beyond the context of geometry, and give non "manageable" examples for teaching, as Zodik & Zaslavsky (2008) comment in a similar case.

CONCLUDING REMARKS

From the seventy six mathematics teachers of our study only thirteen refuted correctly the invalid claim of student A, eleven by constructing a counterexample and two by using theorems. Some of the characteristics of teachers' reasoning that were identified in this study are similar to those reported by Lin (2005) in the case of students. For example, as in Lin's study, there were teachers who confirmed the invalid claim, some who used rhetorical arguments, others who gave the possibility of a counterexample without generating it and few who constructed a counterexample accompanied by a mathematical proof. In our study, teachers' personal example space seemed also to be the base for their generation of counterexamples. However, random examples could not be appropriate in this case as the problem of existence of the triangles was crucial. The counter example in this case is only one and it is generic (Mason and Pimm, 1984) in the sense that it can lead to a general theorem. So, in this particular task the classification of counterexamples in terms of their explanatory power (Peled & Zaslavsky, 1997) has a different meaning. Another issue that emerged from our study was the way that a number of teachers conceived theorems in their argumentation. They believed that a claim can be refuted when "all" the relevant theorems that are included in the school textbooks cannot be applied. This conception indicates a misleading epistemological view of the theorems and their role in mathematical reasoning.

In this paper, we demonstrated teachers' reasoning for refuting students' claims in respect to their mathematical knowledge. Pedagogical issues like the impact of the textbook on teachers' decisions, the role of mathematical communication, as well as students' active involvement were some issues that emerged from our data but have not been discussed in this paper. A further analysis of these issues is needed in order to consider the process of refuting claims in its broad scope.

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"IS THAT A PROOF?": AN EMERGING EXPLANATION FOR WHY STUDENTS DON'T KNOW THEY (JUST ABOUT) HAVE ONE

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This paper describes an episode taken from the third year of a design experiment aimed at improving the teaching and learning of proof at the university level. In the episode, students come enticingly close to having a proof, at least as judged by competent outsiders. However the students themselves, while satisfied with their result, abandon it when asked to write up a formal proof. We offer an analysis of this episode and offer questions for further study.

Key words: Proof, Tertiary Level, Key Ideas, Technical Handle, Design Experiment

INTRODUCTION

Design experiments, or "developmental research" as this work is often called in Europe, are becoming increasingly common, at elementary, secondary, and even tertiary education (e.g. Brown 1992, Collins 1999, van den Akker, Branch, Gustafson, Nieveen, & Plomp, 1999, Lesh 2002). The goal is to find theoretically grounded answers to practical questions of the classroom, done in as natural a setting as possible, with as Brown puts it, the "the blooming, buzzing confusion" that one can sometimes find in real classrooms, under real pressures, with real constraints and opportunities.

While the potential of merging theory and practice is quite alluring for many reasons, the practical and conceptual realities of doing so remain challenging. As Kelly (2002) suggests: if design experiments began in the early 1990's as a sort of art, they are emerging in recent years as a type of science, guided by increasingly rigorous methodology and increasingly useful results. But specifying exactly what this science consists in, that is, how to merge research and practice in a mutually advantageous way, is still a matter of debate, discussion, and development.

This paper is an emerging product of a design experiment aimed at improving the teaching of proof at the university level. The research team, consisting of two mathematics educators and three mathematicians, came together with the aim of improving the teaching of "Introduction to Proof" courses, a type of course used frequently in American universities to help students prepare for the rigor of the theoretical courses like abstract algebra and analysis[1]. The idea was to use videos of students struggling, and eventually succeeding, at proving claims that are known to be hard for students in this type of course, as a basis for discussion. These videos can be used both as a professional development tool for teachers who want to better understand student difficulties with proof and as a curriculum resource for class discussion to help students be more aware about their own mathematical thinking.

After three rounds of testing and piloting, we now have a fairly stable set of curricular materials, which include (1) carefully edited videos of students working on proofs that many other students find difficult, (2) materials to help teachers use these videos, both for their own understanding of student thinking and for classroom use. These materials have been tested in four colleges in the United States in the context of "Introduction to Proof" courses taught by members of the research team and their colleagues. We also are generating a number of research articles, this being one example, that probe questions of mathematical thinking that enhance and/or inhibit proof production.

We consider our particular marriage of theory and practice to be a happy one. The central questions which drive our research—how to reconcile student and faculty thinking about proof and proving—grew naturally from our experiences as teachers struggling to make the best of our own "Introduction to Proof" courses. While not eliminating common sense and experience as legitimate grounds for interpreting data, we felt a real need to move into theoretical territory to help make sense of some of the mysteries of mathematical thinking.

This paper describes one part of this theoretical journey. We begin by describing an episode that our team found particularly compelling. In the episode, students come enticingly close to finding a proof but do not seem to notice that they have done so. Rather than convert what outside observers recognize as a "key idea" of a proof into a formal proof, they abandon the idea and take a different, and ultimately unsuccessful path. This episode is useful for a starting point in understanding the nature of key idea in the process of proof production, but also points to some fuzziness about the notion of key idea, which a more theoretical analysis can help clarify.

FRAMING

The methodology for this project, for which this paper is one small part, follows the program set out by Cobb, et al (2003). The design is highly interactive and interventionist, involving gathering and indexing of longitudinal data from a number of sources, including videos of classroom practice, individual and group interviews with teachers and students, journal and email records from the teachers, written

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[1] See Alcock (2007) for a similar project, focusing primarily on professional development, which has been successfully piloted in the US and UK.

records of student work, and audio and video records of behind-the-scenes discussion among the research team. Like Cobb, et al, we see this design experiment as a "crucible for the generation and testing of theory." It is the tangible pressures of classroom realities that provide a needed spark for the theory to develop and crystallize, and one of the goals of this paper is to make part of that process visible to both research and practitioner communities.

The central research questions involve characterizing the trajectory of proof development in a way that both helps us see where students sometimes go wrong and also gives some guidance towards how to teach students to prove in a more effective way. In particular, as we traced one particular episode in which students struggled, came close, and eventually failed to find a proof we wanted to know (1) what were the critical "moments" when there was opportunity for the proof to move forward, and (2) what is the nature of these moments. In the end we found three such moments, which seem to play a critical role in proof production. These moments do not necessarily occur in every proof, nor do they necessarily occur in the order in which we present them, but they seem to be critical in the sense that if one is present, the proof can move forward in a fairly significant way, and if one is absent, it is quite possible that the proof will not move forward (or that a proof will be produced without a full sense of understanding).

The first moment is the getting of a key idea, an idea that gives a sense of "now I believe it"[2]. The key idea is actually a property of the proof, but psychologically it appears as a property of an individual (we say that a particular person "has a key idea" if it appears that they grasp the key idea of a proof.) We refer to "a" key idea rather than "the" key idea, because it appears that some proofs have more than one key idea. While a key idea engenders a sense of understanding, it does not always provide a clue about how to write up a formal proof.

The second moment, is the discovery of some sort of technical handle, and gives a sense of "now I can prove it," that is, some way to render the ideas behind a proof communicable[3]. The technical handle is sometimes used to communicate a particular

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[2] More elaborated discussions of "key idea" can be found in Raman (2003), Raman & Weber (2006), and Raman & Zandieh (in progress). A key idea can be thought of as a certain kind of intuition that has both a public and private character: public in the sense it can be mapped to a formal proof, private in the sense that it is personally understandable as a sort of primary, or prima facie, experience. For a careful discussion about intuitions see Bealer (1992).

[3] The term "technical handle" here is akin to the term "key insight" in Raman & Weber (2006). We have chosen to change the term in part because it sounded too similar to "key idea" which has a very different character, and in part because the technical aspect of this "moment" seemed central to its nature. The distinction between "key idea" and "technical handle" might appear at first sight to be similar to the distinction Steiner (1978) makes and Hanna (1989) builds on between proofs that

key idea, but it may be based on a different key idea than the one that gives an 'aha'-feeling, or even on some sort of unformed thoughts or intuition (the feeling of 'stumbling upon'.)

The third moment is a culmination of the argument into a standard form, which is a correct proof written with a level of rigor appropriate for the given audience. This task involves, in some sense, logically connecting given information to the conclusion. We assume that for mathematicians the conclusion is probably in mind for most of the proving process. But for students, the theorem might sometimes be lost from sight, adding a sense of confusion to their thinking processes.

In the data below we will illustrate how each of these moments occurs in the midst of proof production before turning more critically to trying to understand the nature of key idea.

THE EPISODE

The following example illustrates the presence and/or absence of these three moments as students work on the following task:

Let n be an integer. Prove that if $n \geq 3$ then $n^3 > (n+1)^2$.

Students were videotaped working on this task in the presence of the research team, and upon their completion, were asked questions about their thinking. Afterwards, the research team watched and discussed the videos. We were drawn to one part of the proof process that turned out to be a genuine mystery—an episode, near the beginning, in which the students generate what the faculty identify as a correct proof, but what the students, at least at some level, do not recognize as one.

Details: In the first two minutes of working on this task, the students made an observation that the professors identified as a key idea of the proof, namely that a cubic function grows faster than a quadratic. Rather than trying to formalize this idea, the students switched to an algebraic approach, what we label as a technical handle, to try to get to a proof. They wrote $n^3 > n^2 + 2n + 1$ which they manipulated into $n(n^2 - n - 2) > 1$ and then $(n-2)(n+1) > 1/n$.

The students then noticed that if $n \geq 3$ then the terms on the left are both positive integers so the product is a positive integer. And since n is an integer greater than 2, the right hand side is going to be between 0 and 1. They wrote these observations as

if $n \geq 3$ (line break) $n-2 > 0$ (line break) $n+1 > 0$ (line break) $0 \leq 1/n \leq 1$

and seemed quite pleased with their reasoning, one student nodding and smiling as the other one wrote the last line.

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demonstrate and proofs that explain. However, it is possible that a key idea gives rise to a proof that demonstrates or explains, and a technical handle can also lead to both kinds of proofs.

S2: Yeah.

S1: This is if n is greater than 3, if n is greater than or equal to 3.

S2: Yeah…. Cool.

At this point in the live proof-writing, the three professors were convinced that the students had a proof. They believed that "all" the students needed was a reordering of their argument. To show $n^3 > n^2 + 2n + 1$, it suffices to show $(n-2)(n+1) > 1/n$, which one can establish by showing that the left-hand side is a positive integer while the right is between 0 and 1.

However, it turned out that the students, despite being pleased with their argument, were less than sure that they were near a formal proof. A professor asked the students "Is that a proof?" and S1 replied, "That's what I'm trying to figure out." As the students moved to now write up the proof, they switched to a new track, trying a proof by contraposition. This attempt ended up turning into a confusing case analysis in which they tried to prove the converse of the contrapositive and investigated many irrelevant cases.

AN EVOLVING EXPLANATION

That students can come so close to a proof without recognizing it is probably familiar to most experienced teachers[4]. *Why* the students are not able to recognize that they are so close is another, more difficult, question. Here we show how looking at the three "moments" of the proof, described above, allows us to compare what the students did in this problem with an idealized version of what faculty might have done.

The moments are represented graphically in Figure 1 below, with the blue line representing the "ideal" (professor-like) proving process, and the red line representing the students' process[5]. The marks m*i* indicate the points in the proof at which different moments are achieved: m1 for the key idea, which both faculty and students achieved (though the students may not realize this), m2 for the technical

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[4] Another example can be found in Schoenfeld (1985) where two geometry students have what the researcher is convinced is a correct "proof" but when asked to write it up, they draw two columns and abandon all their previous work.

[5] In creating this "idealized" version of a proof, we depict a continuity between the key idea and the technical handle, although we realize in practice that many proofs are made without the author being able to connect the two. The question about whether there exists such a connection, even if it has not been found, is an open one. We also realize that the process of proof development is not linear, even for an able mathematician, in many cases. This picture points out more the over-all trajectory of the proof, with minor false-paths ruled out. Further the heights of the peaks could vary.

handle (which students in this case see as disconnected from their key idea), and m3 for the organization of the key idea and/or technical handle into a clear, deductive argument (which in this case the students never reach.)

Specifically, m1 is recognizing that cubic functions grow faster than quadratic ones. m2 is choosing an algebraic approach, factoring the polynomials before and after the inequality sign. We label this as a technical handle even though the students do not know from the beginning where this might lead[6]. m3 is connecting the assumption that $n \geq 3$ with the conclusion that $n^3 > (n+1)^2$. In this case, the students never reached m3, and in fact—during their attempt to write a formally accepted proof, they seem to lose sight of what they are proving.

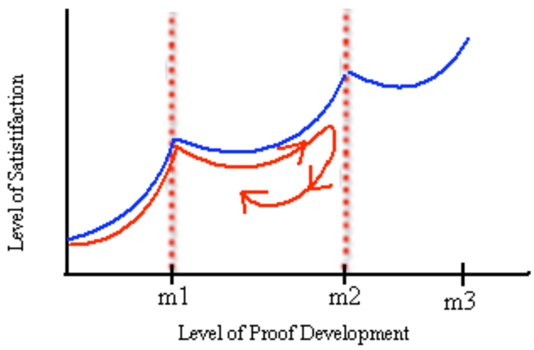

Figure 1: Comparing student (red) and faculty (blue) proof strategies

In the episode above, the students find two key ideas: one that cubics grow faster than quadratics, and another, after students have written $(n-2)(n+1) > 1/n$, that the right-hand term is trapped between 0 and 1 while the left grows indefinitely. Neither of these ideas gets developed into a formal proof. The curved line between m1 and m2 represents how students move towards a technical handle and end up at the second key idea.

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[6] The labeling of technical handle here is a bit tricky. If the students are not themselves aware of the way to link their algebraic manipulation to a proof, is it misleading to say they have found a technical handle since technically they do not seem to register that they "know" how to prove it. We have tentatively labeled this moment as a technical handle anyway, in part because as outside observers we can see that this algebraic manipulation could lead to a correct proof. In addition, while the students might not see exactly how to extract a formal proof from their algebraic arguments, they seem to take their arguments to be convincing and that they have grounds for making a formal argument.

The crucial distinctions between the “ideal” graph and the “student” graph are the breaks at m1 (students do not try to connect their key idea to a technical handle) and m2 (students lose sight of the conclusion and end up trying to prove a converse.) Our data indicate that these breaks are not merely cognitive—it isn’t that the students do not have the mathematical knowledge to write a proof, since they articulate the essence of the proof after three minutes. The problem is epistemological—they don’t seem to understand the geography of the terrain. Expecting discontinuity between a more intuitive argument and a more formal one, the students abandon their near-perfect proof for something that appears to them more acceptable as a formal proof.

Of course it is not always possible to connect key ideas to a technical handle, or to render a technical handle into a complete proof. But what distinguishes the faculty from the students is that the faculty are aware that this connection is possible, and might even be preferable given that sometimes it takes little work—in this case a simple reordering of the algebraic argument would suffice for a proof. As one professor in the study said:

> “It became clear that to formalize meant something different to them and to us. To us, formalize seemed to mean ‘simply clean up the details’. To them, it seemed to mean ‘consider rules of logic and consciously use one’.”

Recognizing the difference between radical jumps that need to be made to move mathematical thinking forward and local jumps that allow one to delicately transform almost rigorous arguments into rigorous ones might be an essential difference that mathematics teachers can learn to recognize, diagnose, and communicate to their students.

FURTHER QUESTIONS

The episode and analysis described above, raise a number of questions which we would like to discuss briefly here.

1. Nature of key idea/technical handle

   One nice feature of the episode above is that the identification of key idea and technical handle came fairly easy, with relatively little debate or discord among members of the research team. But are the notions of key idea and technical handle so clear that they can be picked out in any setting, for any proof? For this we need to continually refine the definitions (and in this paper we have actually backed away from a technical definition and given more general descriptions.) An ongoing research project of our team involves looking at a broad number of theorems, identifying key ideas and technical handles for different proofs, and refining the definitions based on that data.

2. Context of discovery vs. context of justification

The distinction between context of discovery and context of justification[7], which has had a significant influence on epistemology and related fields, might be useful for understanding why students do not realize they have a proof. Taking the distinction to be psychological (which was not the original intent, but serves our purpose here), it seems natural to suggest that in the process of proving one has a phase of discovery and a phase of justification.

In the episode above, the students seem to be missing an important half of this combination. They sort of "discover" the key idea without seeing it as a justification[8]. Perhaps being able to toggle between the different contexts is a marker for mathematical maturity, and somehow central for being able to identify a proof as a proof. Specifically, the key idea might involve some combination of seeing the idea as a product of discovery and a grounds for justification (a thing to be justified). This is just a hypothesis, and a more careful analysis of the distinction between discovery/justification is needed to be able to substantiate it.

3. A Fregean telescope?

Another way of seeing the difference between student and faculty understandings in this episode might have to do with a deep connection (or lack of connection) between mathematical objects and they way they are grasped by the mind. This suggestion is highly tentative: to use Frege's distinction between "sinn" (roughly, sense) and "bedeutung[9]" (roughly, reference) to better understand this relationship (Frege (1892/1997)).

Frege uses the following analogy to explain the difference between sinn and bedeutung: imagine a person looking at the moon through a telescope. The moon is a bedeutung, an object in the world, with a public status. The image on our retina is a sinn, the personal sense we have of that object, which has a private status. The telescope is sort of like a thought that connects the two—it has public status, in the sense that anyone can look through it, but it somehow makes an otherwise difficult to grasp object intelligible to the human mind.

Without going deeply into the way Frege extends this analogy to mathematics (in part because there are tricky moves, both going from the bedeutung of an *object* to the bedeutung of a *sentence*, and going from natural language to

[7] For the original distinction see Reichenbach (1938), and for a critical discussion of this distinction in contemporary philosophy and history of science see Schickore & Steinle (2006).

[8] Wright (2001) warns about misinterpreting the word "discover". He points out that we would not say someone "discovered" the South Pole if they did not realize it was there. It is with this warning in mind that we use the term "discover" in quotation marks.

[9] We retain the German names since the English translations are not completely accurate.

mathematical language) it might be useful to think if there is an analogy to the telescope in the episode described above.

Could it be that students stand facing some (to them) far away star, and with the aid of a telescope the public could be rendered private? If so, what would the telescope be, and is it something that we could better encourage students to develop and/or use as they learn to prove? Or is it possible that there is no telescope at all, just as when I look at the coffee mug on my desk, I feel I am simply getting sense data of the mug, without any mitigation. Perhaps the mind simply grasps key ideas. If so, then, what explains why some people grasp them and others don't?

This is perhaps merely a rephrasing of the central mystery found in the episode above. But by placing this mystery in a Fregean context (which also allows access to his critics), perhaps we gain some conceptual tools to try to better understand, not only the mystery, but also what we can do about it.

These questions mark a few of the places where we think it might be productive to push for a deeper analysis and where we see possibilities to connect to existing research. We are especially excited about the potential to use results from the field of epistemology where questions about the relation between mental representations and the external world (of which we consider mathematics to be a part) have been discussed extensively. In the next phase of our project, we plan to devote increasing time to developing and refining our theoretical ideas. We welcome any and all suggestions that can help us do so.

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DIAGRAMS: A SUPPORT FOR DEDUCTION PROCEDURES

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We present a didactic tool used to aid students in the deduction procedure when writing or understanding proofs. We use different types of diagrams to scaffold their process and provide the necessary elements to identify the theoretical status of a proposition within an axiomatic system and the operational status within a proof.

INTRODUCTION

Teaching and learning to prove are commonly recognized as difficult tasks. Many articulated and coordinated actions are needed to be able to produce a mathematical explanation or proof. One of them is to form connections between the knowledge one possess and given facts, to obtain information that can lead to establishing the truth of a mathematical statement. For the last five years, we have centered our research on different aspects of teaching and learning to prove, and have analyzed some elements that can aid in those processes, like the use of dynamic geometry and the shaping of an environment that propitiates proving activity (Camargo et al., 2007). But only until now have we set our attention on the process of writing a proof and on the two-column format we ask students to use, because, probably, without previous preparation, as Herbst (1999) mentions, its use becomes “the fictitious reconstruction of the production of a proof” and not an instrument to “familiarize them [students] with the meaning of proof in constructing and validating mathematical knowledge”. How else can we engage students with the small dose of formalism and rigor needed to convert an explanation or argumentation into a proof? In this paper we present the diagrams we are beginning to use to prepare students to understand and construct proofs knowingly and to carry out a reflexive and significant use of the two-column format. We also show types of tasks in which the diagrams are used.

DIDACTIC PROBLEM

Constructing a mathematical proof requires, among others, identifying the antecedent and consequent of a conditional; recognizing all the conditions, including the tacit ones, stated in the antecedent and consequent of a theorem, postulate, or definition, to correctly establish their existence among the set of propositions deduced or given, and so decide if a certain fact can be used to deduce information; using valid deduction schemes; assigning the correct theoretic status to a proposition when declaring it as the warrant for a conclusion; and identifying what must be proven.

It is commonly believed that university students reach a skilful level of the above actions as a result of the exposure to their teacher’s, or textbook author’s, proving practice. Research literature about learning to prove indicates that such a belief is a myth; our previous research projects have also shown the necessity of designing and implementing diverse didactic strategies aimed to scaffold our pupils, mathematics pre-service teachers, in attaining mastery of the above actions (Perry et al., 2006).

A figure representation and the use of natural language to communicate ideas are two of the registers needed to construct a plane geometry proof. But, as Duval (2007) points out, a third register is essential to help students organize their ideas and understand how the different propositions fall into place to form a deductive chain that leads from some known facts to a conclusion. We have designed different types of diagrams that become the third register and that differ according to the type of deduction task assigned to students. They are used as a means to indicate relationships between properties; to differentiate the antecedent and consequent of a conditional proposition; to recognize what Duval (2007) designates as the *operative status* of a proposition (i.e., specific role of premise, warrant, and conclusion at a deduction step level); and to stress what Duval (2007) names as *theoretical status* of a proposition (i.e., specific role of definition, postulate, theorem within the axiomatic system of reference). Through the use of these diagrams, students can understand why the theorems, postulates and definitions do prove a statement, and discriminate the status of a proposition from its content, elements Duval considers necessary to learn to prove. "A proof cannot work as a proof as long as there is no comprehension of the specific deductive organization of discourse" (Duval, 2007, p. 159)

DIAGRAMS

The process we follow to aid students in learning to write proofs, used in our geometry courses, is multi-faceted and a means to help make the transition to two-column proofs smooth. We use diagrams in basically four situations. We assign deduction tasks during defining processes and use *definition-diagrams* to illustrate the corresponding reasoning. Whenever a theorem or postulate is established, after an exploration process leads to the formulation of a conjecture, we use a *conditional-diagram* to stress the role played by each part of the conditional statement, and a *deduction-diagram* to stress how the conditional is used to obtain information. This is done before even asking them to intertwine definitions, postulates or theorems to construct a proof, which is when we use the *proof-diagrams*. Putting together these propositions in a deductive chain to establish the truth of a statement is the last stage in the process we guide the students through.

Definition-diagram

Each time a definition is constructed or analyzed in the course, we use the definition-diagram to help students establish the information they can deduce from the definition itself. Hopefully, they will recall the type of diagram used, when they include a definition as warrant in a proof, to correctly assign its theoretical status. In the diagram, the rhombus used to place the name of the object defined and to connect the information placed in the box with that in the oval, points to both sets of information, symbolizes the biconditional nature of a definition. (Figure 1) Even so, within a deductive step of a proof, only one of the included conditional statements is used.

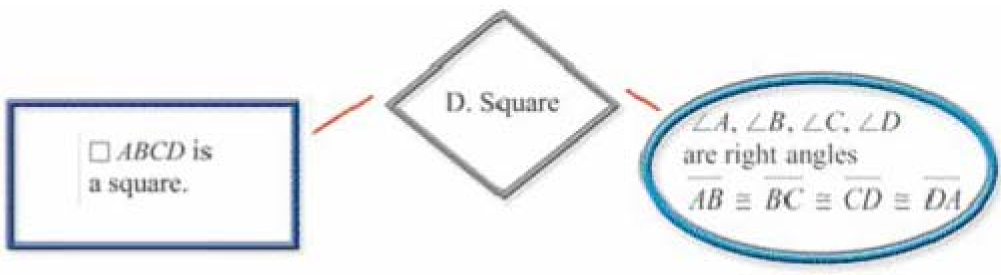
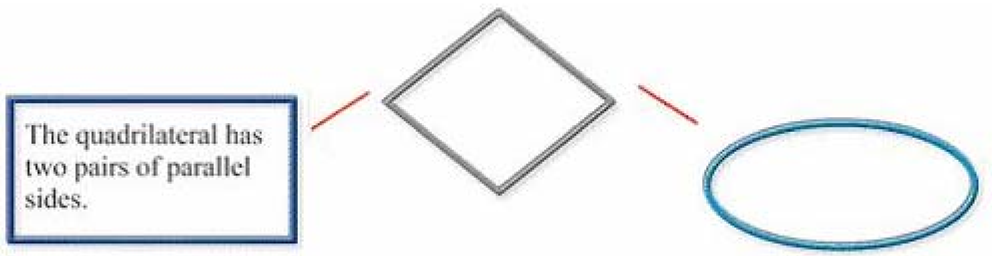


Figure 1: Definition-diagram

Tasks proposed to students with this diagram vary according to the information given. These are:

- Given the defining properties of a figure or of a relation, which can be placed in the box or oval, due to the biconditionality of a definition, the student must identify it by name, or given that a figure is an example of a certain class of figures, the student must make the defining properties explicit. (Figure 2a, 2b)
- Given the name of a figure or relation in the rhombus, the student must particularize the objects involved by assigning names to points and consign the corresponding properties either in the box or the oval. (Figure 2c)

(a)



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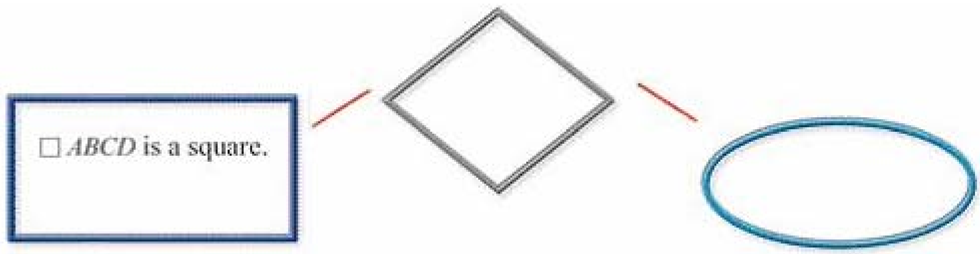


(c)

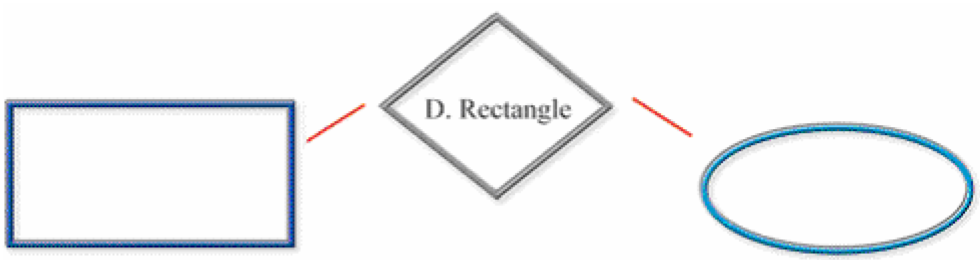


Figure 2: Types of tasks for definition-diagram

Conditional-diagram

In everyday use a tendency to treat conditional statements as if the antecedent and consequent were interchangeable exists, so many students interpret a mathematical statement expressed in the if-then form as equivalent to its reciprocal. This interpretation leads students to actions in which conditional statements produced or used are not in accordance with mathematical procedures or concepts. Examples of these are common illogical reasoning schemes such as stating that the negation of the consequent of a conditional is true if both the conditional and the negation of the antecedent are true. The conditional-diagram, used for theorems or postulates, is designed to help students understand the structure of a conditional statement, to make it meaningful by establishing the dependency relation between the properties involved, and to correctly remember the statement in question. The connecting figure between the two sets of information, where the name of the theorem or postulate is placed, is an arrow that points to the consequent of the conditional, to stress the dependency relation that it enounces (Figure 3).

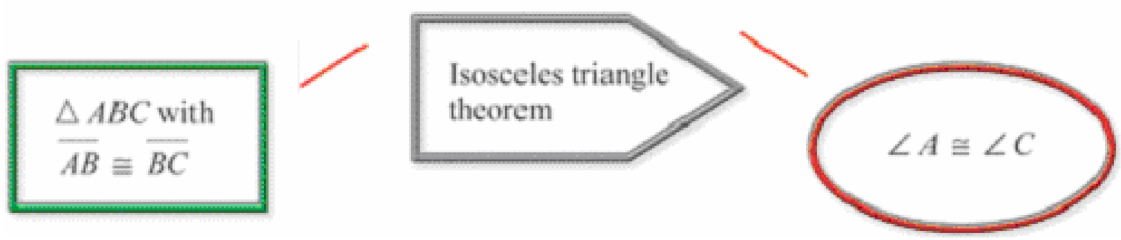


Figure 3: Conditional-diagram

In tasks with this diagram, the premise, conclusion or warrant (theorem or postulate) are placed in their corresponding position. No figural representation is included.

- Given the name of a theorem or postulate, the student must place the antecedent in the box and the consequent in the oval. A student's response will clearly show whether he assumes that the reciprocal is equivalent to the statement and if he is including in the antecedent or consequent all the conditions required. (Figure 4a)

- Given the antecedent of a not designated theorem or postulate, the student must identify it and complete the diagram by placing the corresponding consequent in the oval. The answers are not always unique. (Figures 4b)

- Given the thesis of a theorem or postulate (not designated), the student must identify all the studied theoretic conditional propositions that have such a consequent. Thus plausible reasoning is fostered. (Figure 4c)

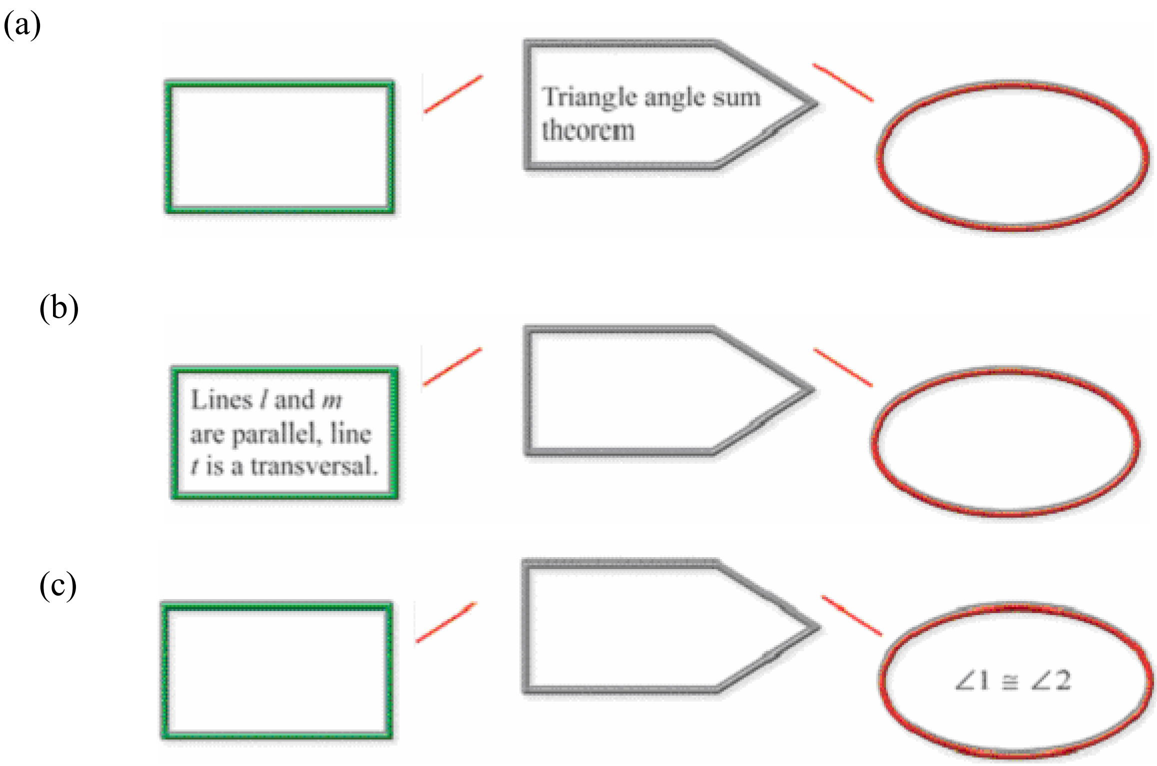


Figure 4: Types of tasks for conditional-diagram

Deduction-diagram

The use of this diagram is solely for a deduction step when the task is to produce information from a given fact; it stresses the role of each component of a conditional in the first organization level of a proof (Duval, 2007). In the diagram, the propositions fall into the corresponding role needed to carry out the reasoning scheme *Modus ponendo ponens* or *Modus tollendo tollens*. It is either the premise ('What I know'), the conclusion ('What I conclude'), or the warrant ('What I use'), that is the theorem, definition or postulate used, the corresponding title emphasizing the status in the deduction step.

What I know	**What I use**	**What I conclude**
$\angle ACB$ is right	D. perpendicular lines	$\overleftrightarrow{AC} \perp \overleftrightarrow{BC}$

Figure 5: Deduction-diagram

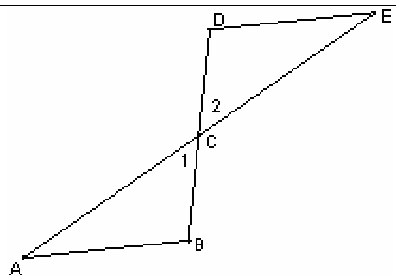
The type of tasks proposed for this diagram are similar to the ones designed for the conditional-diagram but the context is different because now the students must deduce new information. This diagram is established as a link between studying definitions, theorems or postulates as isolated objects to using them in a deductive process to reach a conclusion.

Proof-diagram A

When students are sufficiently familiarized with the three types of diagrams and are ready to produce deductive chains of propositions to prove a statement, the proof-diagram A is introduced, which is the juxtaposition of several deduction-diagrams but where two ingredients are introduced: the hypothesis of the statement to be proved is green and the thesis red, and a different color is assigned to each statement concluded in the proof process. The use of color to distinguish each statement has a specific purpose: the transition from being something deduced to becoming something known that will then play the role of premise in a next step is visually evidenced when the colored statement occupies a different position. This is precisely when the second level, steps organization, of a proof is evidenced (Duval, 2007). Putting together the different colored premises in a 'What I know' cell to use the theoretical element that has them as components of its hypothesis should force students to check if **all** the conditions of the warrant are met. If a figural representation is given, information can be deduced by visual recognition of a property in the figure, when the class socio-mathematics norms have stated that as valid information. That information is consigned in the third column as something concluded. The corresponding warrant is *information given graphically*. Examples of this situation are recognizing betweeness of points, vertical or alternate interior angles, and points in the interior of an angle. That is, in the third column ('What I conclude') all the information that is a product of a reasoning process, either visual or theoretical, is placed.

Example 1. Determine whether the following statement is true. If so, prove it.

If C is the midpoint of $\overline{AE}$ and $\overline{BD}$ then $\overline{AB} \parallel \overline{CD}$.

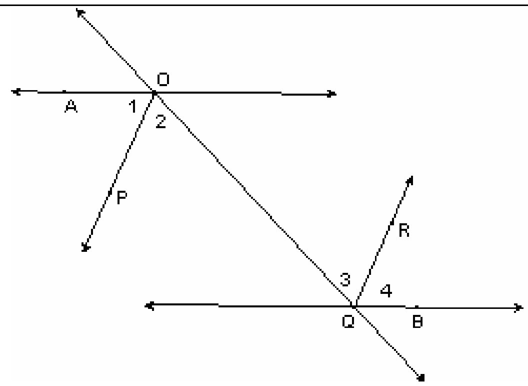


What I know	What I use	What I conclude
1. C is midpoint of $\overline{AE}$	D. midpoint	$\overline{AC} \cong \overline{CE}$
2. C is midpoint of $\overline{BD}$	D. midpoint	$\overline{BC} \cong \overline{CD}$

3.	Information given graphically	$\angle 1$ and $\angle 2$ are vertical angles
4. $\angle 1$ and $\angle 2$ are vertical angles	T. vertical angles are congruent	$\angle 1 \cong \angle 2$
5. $\overline{AC} \cong \overline{CE}$, $\overline{BC} \cong \overline{CD}$, $\angle 1 \cong \angle 2$	P. SAS	$\Delta ACB \cong \Delta ECD$
6. $\Delta ACB \cong \Delta ECD$	D. congruent triangles	$\angle ABC \cong \angle DEC$
7.	Information given graphically	$\angle ABC$ and $\angle EDC$ are alternate interior
8. $\angle ABC$ and $\angle EDC$ are alternate interior, $\angle ABC \cong \angle DEC$	T. Alternate interior angles imply parallel lines	$\overline{AB} \parallel \overline{CD}$

Before students start constructing proofs on their own, we propose exercises as the following one (Example 2), in which the same colored cells must contain the same information. The purpose is to help them identify the proposition that correspond to that deduction step.

Example 2. Fill in the boxes with the information needed to prove that the following proposition is true: If $\overrightarrow{AO} \parallel \overrightarrow{BQ}$, $\overrightarrow{OP}$ bisects $\angle AOQ$ and $\overrightarrow{QR}$ bisects $\angle OQB$ then $\angle 2 \cong \angle 4$.



What I know	What I use	What I conclude
1.	Information given graphically	$\angle AOQ$ and $\angle OQB$ are

		alternate interior angles
2. $\overleftrightarrow{AO} \parallel \overleftrightarrow{BQ}$, ∠AOP and ∠OQB are alternate interior angles		∠AOQ ≅ ∠OQB
3. ∠AOQ ≅ ∠OQB		
4. $\overrightarrow{OP}$ bisects ∠AOQ $\overrightarrow{QR}$ bisects ∠OQB		m∠AOQ = 2m∠2 m∠OQB = 2m∠4
5. m∠AOQ = 2m∠2 m∠OQB = 2m∠4	Transitive property	
6.	Cancellation property	
7.		∠2 ≅ ∠4

Proof-diagram B

The last stage in the process we are describing consists in introducing the proof-diagram B, which is similar to the two-column proof format so commonly used in geometry textbooks, but with a modification. The first and third column information of the deduction-diagram is placed in the column 'Statement' and the warrant in the column 'Justification', which also includes the numbers of the previous steps that contain the premises of the hypothesis of the theorem, postulate or definition used as a warrant. The statements that are given information, hypothesis, are written in green. The propositions that are deduced, either theoretically or from the graph, each have a different colour, and the thesis of the statement being proved is in red. The number that refers to a step has the same colour as the statement of that step.

The following example shows how the written proof of Example 1 is transformed when the proof-diagram is used.

Example 3

Statement	Justification
1. C is midpoint of $\overline{AE}$	1. Hypothesis
2. $\overline{AC} \cong \overline{CE}$	2. D. midpoint (1)
3. C is midpoint of $\overline{BD}$	3. Hypothesis
4. $\overline{BC} \cong \overline{CD}$	4. D. midpoint (3)
5. $\angle 1$ and $\angle 2$ are vertical angles	5. Graphic hypothesis
6. $\angle 1 \cong \angle 2$	6. T. vertical angles are congruent (5)
7. $\Delta ACB \cong \Delta ECD$	7. P. SAS (2), (4), (6)
8. $\angle ABC \cong \angle DEC$	8. D. congruent triangles (7)
9. $\angle ABC$ and $\angle EDC$ are alternate interior	8. Graphic hypothesis
10. $\overline{AB} \parallel \overline{CD}$	9. T. Alternate interior angles imply parallel lines (8), (9)

FINAL REMARKS

We think that the didactic tool here sketched aids students in realizing what is required to be able to construct a proof and what a mathematical proof is. The diagrams are designed to make explicit the way theoretical elements are used in a proof, at each deduction step and in the organization of steps. Color is used to emphasize how the steps are linked to form a deductive chain, as the status of a proposition changes from that concluded to something known. We recognize that the diagrams in themselves have no didactic value without teacher support and the type of exercise proposed to use them. Although we have not carried out systematic research about the effect of the use of these diagrams, we have evidenced that they help remember the status of a proposition as a definition or conditional theorem or postulate, control assuring that the premises of a theoretic element are present before using it as a warrant, and construct a proof. Students enjoy the use of color. We think that the use of the diagrams generates different situations and requirements such as knowing well the different theoretic elements of an axiomatic system. The benefit that the tool produces is also evidenced when the student uses it naturally and shows he understands what the diagram was designed for. They become transparent as they move on to the proof-diagram. This does not mean students do not continue making mistakes in the process of constructing a proof, but any effort teachers make to promote students' understanding proofs and giving them the needed support to learn to prove is never too much. We hope this didactic tool and the actions it generates lead our students to go beyond "a conception of mathematical proof as at most a method for knowledge-certification" (Herbst, 1999) to recognize it as a useful tool for comprehending mathematics and systematizing knowledge.

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"CAN A PROOF AND A COUNTEREXAMPLE COEXIST?" A STUDY OF STUDENTS' CONCEPTIONS ABOUT PROOF*

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In this article we focus on the misconception that a poof and a counterexample for the same mathematical statement can coexist, and we investigate how widespread this misconception is among high-attaining secondary students. The data consisted of 57 student surveys augmented by follow-up interviews with 28 students. While analysis of the survey data alone strongly suggested the existence of the misconception among several students, further analysis with the inclusion of the interview data found no evidence that any of the students held the misconception. Implications for methodology and research are discussed in light of these findings.

INTRODUCTION

Despite the considerable research on students' understanding in the area of proof (e.g., Coe & Ruthven, 1994; Healy & Hoyles, 2000; Küchemann & Hoyles, 2001-03; Sowder & Harel, 1998), students' conceptions about the relationship between proof and refutation have not been the explicit focus of research thus far. Nevertheless, earlier studies (Balacheff, 1988; Fischbein, 1982; Galbraith, 1981; Schoenfeld, 1991) found evidence to suggest that even advanced students possess the misconception that it is possible to have a proof and a counterexample for the same statement. In particular, the studies identified two complementary student conceptions whose combination gives rise to the misconception that a proof and counterexample can coexist: (1) counterexamples do not really 'refute' and (2) proofs do not really 'prove.' Regarding the former conception, studies revealed students have difficulties in understanding that a valid counterexample satisfies the conditions of a conjecture but violates the conclusion (Galbraith, 1981) and documented students' tendency to treat valid counterexamples to conjectures as exceptions that did not affect the conjectures' potential truth (Balacheff, 1988). Regarding the latter conception, studies revealed students' difficulties in understanding that a valid proof confers universal truth of a conjecture thereby making further checks superfluous (Fischbein, 1982) and documented students' tendency to make conjectures that contradicted what they had previously proven (Schoenfeld, 1991).

A possible explanation for the lack of research with an explicit aim to explore the potential student misconception that a proof and a counterexample can coexist is the

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methodological difficulty entailed in designing a research instrument that can be used to elicit students' conceptions in this area. With this article we aim to contribute to this domain of research by discussing a survey that had such a potential and presenting findings we obtained from analysis of student responses to the survey and follow-up interviews. Our work contributes to existing research understandings of students' conceptions about proof by scrutinizing a possible student misconception for which existing studies offered suggestive but incomplete evidence.

BACKGROUND

The research was part of a *design experiment* (see, e.g., Schoenfeld, 2005) that was conducted in two Year 10 classes in a state school in England. The school had 165 Year 10 students (14 to 15 years old) who were set in seven classes according to their performance on a national assessment they took at the end of Year 9. A total of 61 students from the two highest attaining classes participated in the research.

Partially motivated by studies that showed even high-attaining secondary students possessed limited understanding of proof (Coe & Ruthven, 1994; Küchemann & Hoyles, 2001-03), the design experiment aimed to generate theoretical and practical knowledge about possible ways in which classroom instruction can help these students develop their understanding of proof. The design experiment involved development, implementation, and analysis of the effectiveness of a collection of lesson sequences that extended over one to three 45-minute periods. Each lesson sequence was intended to promote issues of proof in the context of mathematical topics and student learning goals that were consistent with the provisions of the English national curriculum, treating proof as a vehicle to mathematical sense making. As far as proof was concerned, the lesson sequences aimed to offer students opportunities to develop their understanding of the limitations of empirical arguments and of the importance of proof in mathematics, to construct proofs for true mathematical statements, and to formulate counterexamples for false mathematical statements. However, the issue of the possible coexistence of a proof and a counterexample for the same statement was not explicitly discussed in the classes.

The definition of proof that guided the work on proof within the two classes was an adapted version of the conceptualization of proof elaborated in Stylianides (2007, pp. 291-300). The following definition was used in the first lesson sequence in each of the two classes as part of students' introduction to the notion of proof.

> An argument that counts as proof [in our class] should satisfy the following criteria:
>
> 1. It can be used to convince not only myself or a friend but also a sceptic. It should not require someone to make a leap of faith (e.g., "This is how it is" or "You need to believe me that this [pattern] will go on forever.")
> 2. It should help someone understand why a statement is true (e.g., why a pattern works the way it does).

3. It should use ideas that our class knows already or is able to understand (e.g., equations, pictures, diagrams).
4. It should contain no errors (e.g., in calculations).
5. It should be clearly presented.

The definition was discussed and referred to by both classes several times during the course of the design experiment, and it can be considered to reflect the classes' 'idealized' shared understanding of the criteria for an argument to qualify as a proof.

METHOD

Data Sources

The data for the article are derived from: (1) 57 student responses to a survey that we administered to the two classes at the end of the third lesson sequence of the design experiment (some students were absent the day we administered the survey), and (2) follow-up interviews with 28 students. The students completed the survey part way through the design experiment, after they had been given learning opportunities to develop understanding of different issues related to proof as described previously.

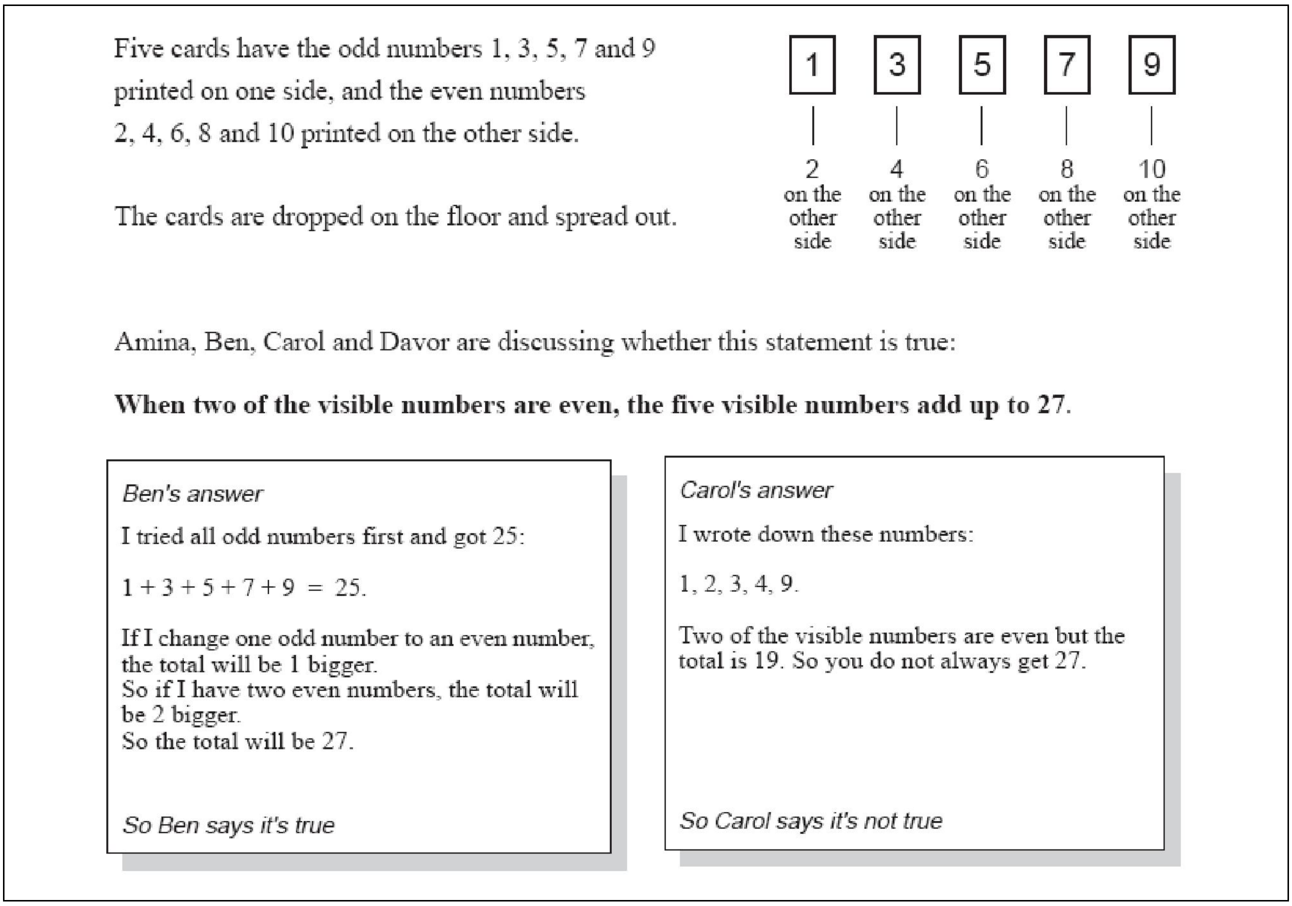

Figure 1: A mathematical problem and two sample solutions to the problem

Survey

The survey presented the students with a true statement contextualized in a mathematical problem, four sample solutions to the problem, and some open-ended and multiple-choice questions (figures 1 and 2); in this article we focus on students' evaluations of only two solutions (Ben's and Carol's).

Open-ended questions:

1. Whose answer is closest to what you would do? Explain your answer.
2. Whose answer would get the **highest** mark from your teacher? Explain your answer.
3. Whose answer would get the **lowest** mark from your teacher? Explain your answer.

Multiple-choice questions:

For each of the following, circle whether you agree, don't know, or disagree.

The statement is:

When two of the visible numbers are even, the five visible numbers add up to 27.

Ben's answer ...	agree	don't know	disagree
shows you that the statement is **always true**	1	2	3
only shows you that the statement is true for **some** examples	1	2	3
shows you **why** the statement is true	1	2	3

Carol's answer ...	agree	don't know	disagree
shows you that the statement is **not true**	1	2	3
shows you **why** the statement is not true	1	2	3

Figure 2: Open-ended and multiple-choice questions.

The survey derived from one used in the Longitudinal Proof Project (Küchemann & Hoyles, 2001-03; Technical Report for the Year 8 Survey, pp. 93-94). We added the third open-ended question and the probes inviting students to explain their answers. We hoped these additions would increase the survey's potential to reveal student thinking about the possible coexistence of a proof and a counterexample. While this issue does not seem to have been one Küchemann and Hoyles aimed to explore (ibid, pp. 6-7), we thought the survey offered an excellent opportunity to do this: some students might not notice the (subtle) mistake in Carol's solution and consider it a valid counterexample to the statement, while at the same time recognize the value of Ben's deductive argument and consider it a proof for the statement.

Interviews

We interviewed 28 students based on their responses to the multiple-choice and open-ended questions in the survey. Most interview sessions began with us asking the students to review their responses to the survey and then to explain which survey question they found the hardest. This general interview probe was followed by specific probes for the students to elaborate on particular responses in their scripts.

Procedure and Analysis

Patterns in students' responses were identified and used to formulate hypotheses about their conceptions. Interview data were then used to test/refine the hypotheses.

With regard to students' conceptions about the coexistence of a proof and a counterexample, our analysis of the survey data focused on those scripts that contained evidence to suggest the potential existence of the misconception. Specifically, we focused on the scripts that contained evidence of one or more of the following 'inconsistencies': (1) the student found a mistake in Carol's solution and said that she would get the lowest mark from the teacher but agreed with the sentence that Carol's solution showed that the statement was not true; (2) the student said that the highest mark from the teacher would go to both Ben's and Carol's solutions; and (3) the student agreed both with the sentence that Ben's solution showed that the statement was always true and with the sentence that Carol's solution showed that the statement was not true.

We coded the type of evidence that was present in the scripts into two categories – strong or weak – depending on the degree of confidence that it gave us as researchers for the existence of the misconception. Specifically, we considered that strong evidence was offered by those scripts that had either "agree" or "don't know" in the first multiple-choice questions for both Ben's and Carol's solutions, and that included no relevant disconfirming evidence in the open-ended questions. The scripts that we considered offered weak evidence for the existence of the misconception had again either "agree" or "don't know" in the first multiple-choice questions for both Ben's and Carol's solutions, but included some relevant disconfirming evidence in the open-ended questions (e.g., they offered evidence that the student was aware that Carol's solution had a mistake). For each of the strong or weak evidence categories we used the interview data to examine the extent to which there was, overall, evidence to suggest that the misconception was indeed possessed by students. Also, we used the interview data to seek possible explanations (from the students' perspective) for the 'inconsistencies' that we identified in their scripts.

RESULTS

General Findings

Our analysis of the survey scripts showed that 16 out of the 28 students interviewed exhibited some evidence to suggest the existence of the misconception that a proof and a counterexample can coexist. Of these, ten scripts showed strong evidence and six showed weak evidence for the misconception. Our subsequent analysis of the interview data revealed that the students in each group (i.e., strong or weak evidence group) tended to offer similar justifications for their choices.

Regarding the strong evidence group, our interview data suggested that the inconsistencies in students' responses derived from them considering Ben's and

Carol's solutions in isolation from one another when they were completing the survey. While discussing their responses with the interviewers, however, all the students in this group became aware of the potential inconsistency between their evaluations of Ben's and Carol's solutions, presumably because the interviewers' questions directed (implicitly or explicitly) students' attention to the relationship between their evaluations. Yet the manner in which the students became aware of this inconsistency and how the awareness played out in the interviews varied.

On the one hand, some students realized the mistake in Carol's solution without any prompting from the interviewers and immediately dismissed her solution. As a result of this dismissal, there was no opportunity for the interviewers to explore further whether these students would experience any sense of conflict that a proof and a counterexample can coexist. On the other hand, some other students needed explicit prompting from the interviewers to reflect on whether or how their evaluations of Ben's and Carol's fitted together before they appreciated the potential inconsistency between these evaluations. Believing that Carol had found a genuine counterexample, these students attempted to resolve the emerging conflict by assuming there was a flaw in Ben's argument, which however they were unable to identify. The interviewers then helped these students see the mistake in Carol's solution and realize it was not a genuine counterexample. As a result of this realization, the students subsequently rejected Carol's solution, but this rejection was not always accompanied with endorsement of Ben's solution as a proof.

Regarding the weak evidence group, our interview data suggested that the students in this group seemed to be aware of the following 'inconsistency' we identified in their scripts: the students pointed out the mistake in Carol's solution in their response to the open-ended questions, but in the first multiple-choice question for Carol's solution they agreed that the solution showed the statement was not true.

During the interviews, the students argued, with different degrees of clarity, that, in spite of the mistake in Carol's solution, her reasoning should be valued because her logic was correct and she had disproved a statement, albeit a different one from that in the problem. Consequently, none of these students changed their minds about their evaluations of Carol's solution during the interview. The issue of the misconception was not pursued further by the interviewers, as the students were already aware that Carol's argument was not a counterexample to the particular statement.

To sum up, there is no evidence from our interviews to suggest that any of the 16 students we originally identified as potentially having the misconception actually had it. Furthermore, the interview data showed that any potential conclusions that could be drawn from the survey data alone would be insecure, as students appeared to have good reasons for 'inconsistencies' we identified in their scripts. For this reason we do not report findings with students we did not interview.

Illustrative Case 1: The Case of Emily

The first case is of a student we call Emily, whose responses to the survey showed strong evidence of the misconception. Emily's script had "agree" in the first multiple-choice question for Carol's solution and "don't know" in the corresponding question for Ben's solution. Furthermore, in response to the second open-ended question, Emily wrote that both Ben and Carol would receive the highest marks from the teacher and justified her thinking as follows:

> Ben: It [Ben's solution] is carefully thought out and written down in an understandable and clear manner.
>
> Carol: She has shown when it [the statement] is not true.

During the interview Emily explained her thinking about Carol's solution as follows:

> The question was saying [that] when two of them [the visible numbers on the cards] were even that the answer is always 27, but she proved that it's not, so she answered the question that was being asked.

In regard to Ben's solution, Emily said:

> It [Ben's answer] was very, like, well set out and easy to understand and I think that was how I would have done it cause the other answers are like gabbling on a bit and they don't really explain why it's [the statement is] true or false.

She explained further that her "don't know" response in the first multiple-choice question for Ben's solution was because Ben "didn't show that it's always true, he only showed it for some numbers." When asked whether she thought Ben had a proof, Emily said that Ben "needed to maybe expand it [his solution] a bit more to convince people that it was true" and noted that Ben could come up with a proof if he worked a bit harder on his solution.

After summarizing what Emily said about the two arguments, the interviewer asked Emily how her two evaluations fitted together. Realizing the inconsistency between the evaluations, Emily laughed and said: "they don't [fit together] because Carol's proved that it's wrong and so it's impossible to prove that it's true… cause it's not true!" Asked what she thought was going on with the two arguments, Emily asserted:

> They [Ben and Carol] have both tried different ways and got different answers, so if they kept working at it, if Ben kept working on his [solution], he would eventually figure out that it's not true.

The interviewer then helped Emily to see the mistake in Carol's solution. Once Emily realized the mistake, she exclaimed: "Oh, so she [Carol] could be wrong… so hers is wrong then." On reviewing her original responses to the multiple-choice questions for Carol's solution, Emily decided to change her response to the first question from "agree" to "disagree," because, as she said, Carol "hasn't followed the instruction." Emily concluded that Ben's solution "might be true" but she decided not to change her responses to the multiple-choice questions for his solution.

Illustrative Case 2: The Case of Evans

The second case is of a student we call Evans, whose responses to the survey showed weak evidence of the misconception. Evans' script had "agree" in the first multiple-choice questions for both Ben's and Carol's solutions, an indication of the existence of the misconception. Furthermore, Evans' responses to the first two open-ended questions showed particular appreciation of Ben's solution: he wrote that Ben's solution would be close to what he would do and that the solution would get the highest mark from his teacher "[b]ecause [it] shows working and offers convincing proof." Yet Evans' response to the third open-ended question offered disconfirming evidence of the existence of the misconception as it indicated that he was aware of the mistake in Carol's solution and said that Carol's answer would get the lowest mark from the teacher. In a series of two interviews, we tried to understand the reasoning for the apparent contradiction in Evans' evaluation of Carol's solution.

Evans was aware that Carol's solution had a mistake in it, but on the basis that she applied a correct mathematical method and that this application warranted recognition, he consciously agreed that she had shown the statement was not true.

> Well what she [Carol] has done is like impossible because 1 and 2 can't be seen at the same time, so then I would have disagreed because that can't be true. But seeing as though she has shown that she's thought it through and like, with her own reasoning she's come to an answer, then I would have put she technically has [shown the statement is not true] but she's got it wrong. […] Carol tried to prove the statement wrong, so one counterexample was enough. She had the logic right but she didn't succeed to come up with a correct counterexample.

This interview excerpt that Evans evaluated Carol's solution from her own point of view and that he understood the fundamental idea that a single counterexample suffices to refute a general statement. Evans considered that Carol's solution embodied understanding of the latter idea, even though the counterexample that she offered did not satisfy, as he observed, the problem's conditions.

When pressed by the interviewer to explain his thinking further, Evans described the different evaluation standards that he perceived existed in exams and in class work:

> In an exam you don't get marks for the proof, do you? You get marks for showing your working and actually getting the answer in the end. But it [Carol's solution] does show the proof and everything. I don't know, it depends on what sort of question it is… if it's like what we're doing proof and stuff [referring to the proof work in class] then that [Carol's solution] would probably get the highest mark if that was what it was marked on… but in the exam it would be marked differently because it's not about how you are thinking, it's about getting the answer and getting the working and everything right.

The interviewer did not raise explicitly the issue of the possible coexistence of a proof and a counterexample, as Evans was clearly aware that Carol's argument was not a valid counterexample to the particular statement in the problem.

DISCUSSION

Although several student scripts offered considerable evidence for the potential existence of the misconception that a proof and a counterexample can coexist, our analysis of the interview data found no evidence of the misconception. This ‘mismatch’ might have been influenced in part by what we considered as evidence (weak or strong) for the possible existence of the misconception in our analysis of the survey data: stricter definitions could reduce the number of scripts that contained evidence to suggest potential existence of the misconception, thereby reducing the gap between the tentative conclusions drawn from the survey data alone and our conclusions with the inclusion of the interview data. However, the mismatch is perhaps illustrative of a broader methodological phenomenon, namely, that student responses to surveys might, by themselves, offer a rather limited insight into students’ conceptions and that follow-up interviews with selected students are necessary for construction of a trustworthy picture of students’ conceptions. Indeed, as exemplified by the cases of Emily and Evans, a wealth of information about students’ conceptions about proof was gained from our interviews with the students based on their responses to the survey.

Despite the aforementioned limitations in the conclusions that could be drawn based on the survey data alone, the survey offered (as we anticipated) a meaningful context in which we raised and discussed with students their ideas about the possible coexistence of a proof and a counterexample. Carol’s argument worked particularly well for the purposes of our research, as the subtle mistake it included in the violation of the problem’s conditions passed unnoticed by several students, thereby helping us meet the methodological challenge of presenting students with a believable ‘counterexample’ to a true statement. Ben’s argument did not work as well: students like Emily recognized the value of Ben’s argument but did not accept it as a proof, primarily because they thought it needed ‘unpacking.’ The fact that the students did not consider Ben’s argument a proof gave them an easy way to resolve the problematic situation we tried to create for them regarding the possible coexistence of a proof and a counterexample: students suspected a mistake in Ben’s argument and endorsed Emily’s counterexample. Given that the statement in the problem that we presented to the students was in fact true, it should not be hard to refine Ben’s argument in the survey to make it an argument that more students can accept as a proof. This revision can increase the survey’s potential to elicit students’ conceptions about the possible coexistence of a proof and a counterexample.

The fact that our research did not reveal this misconception does not mean that there are no students who have it; less advanced students, younger students, or students with fewer experiences with proof are more likely to have it than the students who participated in our research. The absence of the misconception in our data challenges, however, the suggestive evidence offered for the misconception by the findings of earlier studies as we explained in the introduction and raises the need for further scrutiny of this and perhaps other (mis)conceptions that we (mathematics educators) tend to attribute to students in the area of proof.

Finally, the difficulty of some students to recognize Ben's argument as a proof may be indicative of the fact that it is generally harder to validate a purported proof than a purported counterexample, presumably because a proof is often a more complicated argument than a counterexample that draws on a wider range of valid modes of argumentation. The difficulty entailed in the validation of a purported proof may explain in part that even advanced secondary students wanted to check more examples of a statement for which they had already accepted a proof (Fischbein, 1982): perhaps the students were searching for possible counterexamples and their acceptance of the proof was not as firm as they communicated to the researcher.

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APPROACHING PROOF IN SCHOOL: FROM GUIDED CONJECTURING AND PROVING TO A STORY OF PROOF CONSTRUCTION

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This paper presents some aspects of an ongoing research aimed at leading students (through activities of conjecturing, guided construction of proof and story making of the rationale of the proof) to become aware of some salient features of proving and theorems. Theoretical elaboration as well as an example of didactic engineering concerning Pythagoras' theorem will be outlined.

I INTRODUCTION

School approach to theorems has been a subject of major concern for mathematics education in the last two decades. Students' learning to produce proofs and their understanding of what does proof consist in (Balacheff, 1987) have been considered under different perspectives and with different aims: among them, how to make students aware of the differences between proof and ordinary argumentation (Duval, 1991, 2007); how exploit "cognitive unity" (which for some theorems allows students to exploit the arguments produced in the conjecturing phase to construct the proof) in order to smooth the school approach to theorems (Boero, Garuti & Lemut, 2007); in what cases of cognitive unity do students meet difficulties in the passage from an inductive or abductive reasoning, to the deductive organization of arguments (lack of structural continuity: Pedemonte, 2007, 2008); what are the common aspects between ordinary argumentation and proving, and how to prepare students to proving by relying on those aspects (Douek, 1999a, 1999b; Boero, Douek & Ferrari, 2008).

Previous research work helps us to formulate and situate some educational problems that arise in the school approach to theorems: how to tackle theorems for which cognitive unity does not work, or (if cognitive unity can work) when students meet important difficulties due to the lack of structural continuity? How to make the students aware of some salient characters of proving and proof? and how to lead them into some specific competencies of proving activity? In this paper we propose a possible way to tackle these problems in an integrated way. The idea is to guide students' constructive work on proving, then to help themfocusing on the characteristics of the organisation of proof.

This paper presents a theoretical and pragmatic elaboration about how to deal with theorems for which cognitive unity does not work, and approach the rationale of a proof at first stages of proof teaching and learning. The theoretical elaboration also frames the accompaniment of students through two aspects of proving activity: exploration (either in order to find a statement, or in order to find reasons for validity of a statement); and organisation of reasons (or arguments), in the perspective of producing a proof, or in the perspective of understanding the links between statements or arguments in a proof. Our hypothesis is that the rationale of a proof can be approached early in the school context through “story making” situations, preceded by suitable activities of conjecturing and guided proof construction, and related classroom discussions. We illustrate this position by an example of a didactical engineering concerning Pythagoras' theorem.

II FRAMING PROOF CONSTRUCTION

Inspired by Lolli's analysis of proof production (see Arzarello, 2007), we consider proving as a cognitive and a socio-culturally situated activity engaging four modes of reasoning.

1. Heuristic exploration. It occurs when one tries to interpret a proposition or to produce a proposition or an example. One has in mind some target but the main focus is not on attaining the target through an acceptable mathematical reasoning. Any accidental event, writing, metaphor, may move the exploration activity. This type of reasoning is typically open to divergent paths.
2. Organisation of reasoning, making explicit the threads of reasoning holding propositions together. When a proposition seems pertinent, a calculation promising, a writing efficient, one searches for a convincing coherent link to a local goal or to the global one. It is also so when trying to understand a proposition or a reasoning. The links may be theoretical reasons of validity. They may remain at the level of analogy or metaphor, relating goal to steps, at least for understanding. The intentional and planifying characters are typical of this mode, and abduction is a good example of it. Mathematical deductive reasoning is not yet a priority. Such organisation intention may concern partial arguments or the whole of the argumentation aimed at proof construction.
3. Production of a deductive text following mathematicians' norms. Once ideas of proof are brought to light, they must be organised in a deductive reasoning.
4. Formal structuring of the text, to approach a formal derivation. This mode will not at all be approached in the school context we are considering.

These four modes could be considered as successive phases of a proof construction, as different moments with different intentions. But in fact, as reasoning modes, they seldom do appear separately. Not only the succession of modes can vary and loop, but even two or more of them may intervene very closely in one phase aiming mostly at exploring or at writing a deductive text, for instance. Jumps from one mode to another can occur at each stage of the proof production process. Example of loops in the succession of phases: once one thinks she has produced a correct deductive text and moves towards a more formal structure (evolving from Mode 3 to 4), she may need to explore compositions of elementary propositions varying quantifiers, negation, etc... to find the right logical proposition: she returns to reason in mode 1 upon the formal structuring. Or the formal constraints make her discover that some non trivial conditions of validity have not been considered, so she reasons in mode 2 to find the arguments that allow these unexpected conditions to be satisfied. Example with intertwined modes of reasoning: when exploration (mode 1) leads to an interesting idea, examining its efficiency, developing an abductive reasoning to find out a well related theorem or procedure (mode 2) or a mathematical deductive reasoning to consider its consequences in the perspective of the goal (mode 3) are locally produced to evaluate plausibility and connections before returning to explore some other idea. In this second example, the different reasoning modes can be intertwined within one phase of the proving process that would rather be characterised as exploration, just because it would be mainly so in the intention of the subject. Such quick jumps are not conscious and do not need to be, but we make the hypothesis that they are acquired through conscious jumps between clearly characterized phases.

Methodologically we could say that the phases are didactical tools to organise teaching-learning situations into sequences with clear didactical goals, the modes being cognitive tools for analysing student's activity. Thus we refer to phases through predominant modes of reasoning. As a matter of fact, a didactical goal is to lead the students to be aware of the processes they have to go through, essentially to favour students openness in exploration and their rational control in organising reasoning. But no exploration is blind nor any reasoning organising is totally controlled, which means that when we analyse a phase of exploration activity, we ought to capture some reasoning organising activity, etc... (see sequence 1, in V).

The different modes and phases of reasoning involve several cultural rules of validity, and they affect the delicate game of changes from what is allowed or even needed for one mode, to what is allowed or needed for another. For instance abductive reasoning is typical of mode 2 but is not allowed in modes 3 and 4, and student will have to move from it towards deductive reasoning, which is not easy. We can also consider the use of examples (pertinent in mode 1 and 2 but not acceptable in modes 3 and 4), and the conscious handling and conversion of different semiotic registers according to different modes of reasoning (Morselli, 2007; Boero, Douek & Ferrari, 2008).

This analysis leads us to give a special role to argumentation both as an intrinsic component of reasoning, and as a didactical tool to manage the different modes of reasoning and the relationships between them in a conscious way, keeping into account specific cultural rules (to be mediated by the teacher).

III ARGUMENTATION IN PROOF AND PROVING

In this paper, an "Argument" will be "A reason or reasons offered for or against a proposition, opinion or measure" (Webster), including verbal arguments, numerical data, drawings, etc. An "Argumentation" consists of one or more logically connected "arguments". Proof itself is an argumentation. But other argumentations play an important role in proving. Mode 2 is specially based on intentional argumentative activity: discussing the use of a theory or a mathematical frame to produce a step of reasoning relies on a meta-mathematical argumentation (Morselli, 2007). It is not really part of a proof, but is needed to produce it. Metaphors and analogies may implicitly affect mode 1 reasoning or be explicit arguments in mode 2 (Douek, 1999a, 1999b).

For teaching and learning purposes, argumentation is a fruitful means to control the validity of reasoning (as the legitimate use of examples and, or transitions from one mode of reasoning to another with their different cultural rules).

We are therefore interested in two levels of argumentation: as part of the proving tasks, specially for producing and organising arguments (mode 2); and in discussing procedures, as a means to assimilate and master elements of proving processes.

Convergent structure of argumentation in a proof

In general, an argumentation is made of more elementary ones that may be organised in various ways (converging towards a conclusion, or being parallel as when producing different explanations, etc.). In a proof, the elementary argumentations may form a linear chain, each conclusion being input as an argument for the following argumentation, thus forming one whole "line of argumentation". But in many cases of proof, argumentation may contain parentheses "blocks", or side argumentation branches that meet the main line to input a supplementary data or argument. A parenthesis might be considered as a secondary line of argumentation. This description underlines the possible hierarchical relations between various argumentations involved in a proof (Knipping, 2008), which is a difficult matter for students who are being introduced to proof (see the Example for a suggestion).

IV EDUCATIONAL ASPECTS

In the early stages of proof teaching and learning, students can be smoothly introduced to theorems and proofs by conjecturing and proving activities provided that cognitive unity works (Boero, Garuti & Lemut, 2007). In particular exploratory activity (Mode1) and justification (Mode 2) can be introduced at early stages. In a suitable educational environment, 7th and 8th graders are able to produce conjectures for non trivial arithmetic or geometric situations, and move (under a loose guidance by the teacher) towards constructing general justifications. Comparison of students' productions and classroom discussions about them, orchestrated by the teacher (Bartolini Bussi, 1996; Bartolini Bussi & al, 1999) allow students to appreciate some relevant cultural requirements of conjectures and proofs, like their generality and the conditionality of statements (Boero, Garuti & Lemut, 2007), and to become aware of processes favouring conjecturing and proving.

In the following we will focus on mode 2 reasoning, specially in the organisation of reasoning phases, then in the didactical engineering we will also consider mode1 more specially related to conjecturing.

In spite of their usefulness to initiate students into conjecturing and proving, in those cases in which cognitive unity works well, with no difficulties due to lack of structural continuity (Pedemonte, 2007, 2008), the peculiar argumentative structure of a proof does not emerge as an object of reflection for students. Indeed both the facts that easy-to-prove theorems must be proposed for a smooth approach to theorems and that the students themselves are able to enchain the arguments in an autonomous way, make artificial and rather empty the discussion about the specific argumentative arrangement of those arguments.

However students must be enabled to move from theorems for which cognitive unity works to theorems (like Pythagoras') for which proof cannot consist in the deductive arrangement of arguments produced by conjecturing. For other theorems students can meet difficulties in moving from creative ways of thinking (abduction, induction) typical of conjecturing to deductive arrangement of the produced arguments (Pedemonte, 2007). In both cases proving needs a strongly guided activity; and teachers' guidance can even initiate students' awareness of the mechanisms inherent in the Mode 2 reasoning, and open the perspective of Mode 3. Drawing from theoretical reflections, we make the hypothesis that the inherent argumentative activities could be promoted through debates (with real others) about arguments and their relations on one side, and story making on the other.

The debate

Classroom debates, if well oriented and guided, stimulate efforts of expression and explanation. These efforts, in turn, favour the consciousness of the logical rules and their range of validity. For instance, discussing a statement may bring students to understand that producing an example to support it can be an efficient step in the exploratory phase, but is not a valid argument when organizing a general mathematical justification. It may drive them to understand that some semiotic registers (like drawing) are crucial for exploration, and may be for organising reasoning, but insufficient to produce a suitable argument in a deductive reasoning. R Such discussions question cultural rules of mathematical reasoning and mathematical knowledge too. Also the relation between arguments and the construction of lines of argumentation (mode 2) can be discussed in a debate, which draws students' attention to the goal of the line of argumentation in relation to its steps.

Making a story

> Logic is concerned not with the manner of our inferring, or with questions of technique: its primary business is a retrospective, justificatory one - with the arguments we can put forward afterwards to make good our claim that the conclusions arrived at are acceptable because justifiable conclusions.

This quotation from Toulmin (1974, p. 6) inspires the hypothesis that in order to grasp the rationale of a proof, students may make an individual story from the ideas and calculations involved in a reasoning that validates the statement. We emphasise the story that connects steps and fragments with reasons, in order to serve the conclusion, and not particularly the story of how the steps occurred in one's mind (Bonaffé, 1993), nor of how learning has evolved through time (Assude & Paquelier 2005). The goal is that students recognise the involved lines of argumentation, their possible hierarchical relations, and their role in the logical combination that produces the proof. At least at first stages of proof learning, these individual story makings need to be prepared by suitable tasks of guided construction of proof and by related debates putting into evidence some crucial "steps" of Mode 2 reasoning.

In our theoretical construction, debates and story makings should be considered together and arranged as a dynamic system of complementary situations. Individual story making involves students in an active personal reconstruction of the rationale of a proof, while a debate on the work done in individual tasks of conjecturing and guided proving (and story making as well) offers both openness to other possible combinations and regulation. We expect this system to draw students' attention to the "components" of the story. The deductive structure of the proof (through mode3) will consist of a particular relating of the pertinent components of a story.

Students need to be gradually initiated in both activities, possibly before the activities on theorems in order to establish a suitable didactical contract (Brousseasu, 1986). However story making, in the case of theorems, shows particularities that need a careful mediation through sequencing suitable tasks.

Before illustrating the above theoretical reflection by an example, let us present the main activities we wish students to develop and their co-ordination.

- Associate exploration and conjecture to enhance mode 1 (without excluding other modes).
- Stimulate proving the conjecture(s); either the cognitive unity can work and thus the students are able to produce a proof, or it cannot work and the teacher offers a task to guide them towards the proof.
- Engaging the dynamic system of collective debate / individual story-making, starting from discussing some of students' productions, to enhance mode 2 (without excluding other modes). In case cognitive unity could not work, students would not be in a good condition to understand the proof nor to learn much of it, and this dynamic becomes particularly crucial.

V AN EXAMPLE CONCERNING PYTHAGORAS THEOREM

Pythagoras theorem was chosen for two reasons: it is an important and early met theorem in school mathematics; and it is not difficult to get the conjecture through a loosely guided path, while the construction of a proof needs a strong guidance by the teacher (cognitive unity cannot work, because the geometric constructions needed for the usual proofs are not suggested by the work done in conjecturing). Teachers' guidance, classroom discussions and story making will allow the students to approach the rationale of the proof and offer occasions for learning about proof and proving.

First sequence: “Discovering” Pythagora's theorem, expressing the conjecture and making sense of it

Students have not only to grasp the theorem, but also to develop some proving skills (though no proving activity is demanded in this phase) and prepare for the further work; thus the activity on Pythagoras' theorem is prepared by Task 1 (an individual production on another theorem), followed by classroom discussion:

> Task1: Consider the statement: "In a triangle of sides a, b and c, a+b is always smaller than c". Is it true? always? Why? Prepare yourself to explain how you checked it and why you think it is true, or it is not, or what makes you doubt.

No triangle is presented by the teacher; students are encouraged to draw some triangles for a check, if they did not do it spontaneously. This task aims at exploration through testing examples, and (specially in the discussion) at leading the students to express the rationale of the activity and to make visible the generality of the proposition they produce. An expression like “we wanted to see if it is true that... so we tried to verify it with four examples” is encouraged: such simple story making reflects an ability (and invites) to reconstruct the logical skeleton of the activity they went through. It bridges a Mode 1 reasoning with a Mode 2, and prepares Task 2.

> Task 2 (individual): Now if we consider the squares of the lengths, instead of the lengths themselves, the situation is different. See if a relation between the squares of the lengths of the sides of a triangle exists. Once you think you produced a valid statement (a "conjecture"), put it clearly in words to explain it to other students.

Right angled, acute and obtuse triangles, are presented on the worksheet. Afterwards a collective discussion guided by the teacher is engaged to share and discuss the conjecture(s) produced, and the ways followed to produce them; and to attain and share acceptable expressions of the conjecture(s) (according to mathematical standards). An incomplete conjecture or an erroneous one may offer fine opportunities to make explicit the important elements of the theorem (in particular the condition of validity of Pythagoras' theorem, i.e. the angle being right) and their role.

> Task 3 (individual): Write down the conjecture as now you think it should be. Explain it and illustrate it with some examples.

The teacher concludes with the standard formulation of Pythagoras' theorem.

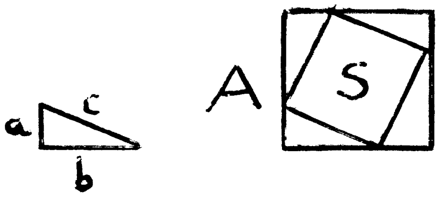
Concerning proof learning, this first sequence aims at involving students in Modes 1 and 2: Exploring (drawing, measuring, calculating, induction when modelling and producing algebraic expressions, repeating procedures and modifying data) mostly in mode 1; and, mostly in mode 2, organising the exploitation of the gathered data, classifying them in order to find some rule, expressing results as general and in rather conventional way (in everyday language is acceptable), etc; discussing and justifying propositions, and organising the steps of exploration in relation to a goal.

Note that classifying and modelling are as much in mode 1 and mode 2. The explicit intentions of exploration and of organization are satisfying sign in my opinion, as a main didactical goal is to enhance the processes students have to go through.

Second sequence: Guiding Pythagoras' theorem proof, and teaching/learning to organise the steps of reasoning into lines of argumentation

Given that cognitive unity cannot work, students are guided through the proof within individual and collective activities; then they reconstruct the lines of argumentation.

> Task 4 (individual): Here we study the proof of the theorem we have conjectured, you will be guided towards this proof. Consider a right angled triangle with sides a, b, c. We use it to build the square A (see below). Its central square S is of area c^2.



> I) Can you explain why S is a square (of area c^2)?
>
> II) Try to write the area of A in two different ways. Find and explain the two ways.
>
> III) How can this help us to validate our conjecture?

A geometrical reasoning is expected to intertwine with an algebraic reasoning in order to attain the equality between the areas. If needed, some supplementary tasks can be inserted either for the whole group or for some students.

After students' individual work, the teacher orchestrates a discussion concerning the reasoning that allows to prove the steps of argumentation and the calculations and why they are needed, and in particular, the connection between geometrical arguments and algebraic arguments.

Task 4 is formulated and organised in a way to approach a story making of the proof. The subsequent discussion of the organisation of the lines of argumentation and the insertion of "blocks" of arguments/calculations in the main line is meant to prepare students to write a “story”.

Third sequence: story making

> Task 5 (individual): Write down how you organized your steps of reasoning to reach a general justification of the conjecture, and justify why those steps are important

This task is particularly important for those students who were not productive in the previous sequence in order to allow them (as well as the other students) to grasp and reconstruct the rationale of the proof.

Here is the kind of arguments we hope the students produce:

> first *(block 1)* we calculate the area of A, then *(block 2)* we find another algebraic

expression of it, because *(looking forward to the final goal)* surface measures of squares are written as algebraic squares. So we think that a^2, b^2 and c^2 will appear and will be related *(possibility to rejoin the main line)*. So, we can write the algebraic equality, and find the relation after transformations.

Mode 2 reasoning is needed for this task in block 2: students must go through an abductive reasoning ("how can I find a^2 and b^2 in this big square?") while deduction prevails in block 1 and will prevail afterwards, till the end.

It is important to notice with the students that the algebraic equality is the principal aim (and first to come to the mind, since it is near to the conclusion we want to reach) but that we have to begin with geometrical considerations, which are like parentheses besides the principal aim. Thus the reasoning is made of a principal line of argumentation and side parentheses involving geometrical reasoning and calculations, whose conclusions flow into the main argumentation line.

Getting familiar with mathematical proof practices (like moving from a geometrical frame to an algebraic one, using geometry only for strategic purposes...) is a particular aspect of this work.

Difficulties inherent in the classroom implementation of the proposal

Comparing the proposal with the style of teaching of most teachers, and keeping into account my first experiences of work with teachers on this subject, I must say that teachers meet some difficulties in engaging in a coherent classroom implementation of the proposal. One difficulty consists in the fact that "To produce a conjecture" is a task that does not fit the most frequent didactical contract in our schools (statements are usually presented and illustrated by the teacher, and learnt by students who repeat and apply them afterwards; the same for proofs). Another is that teachers tend to identify student's task of reasoning and the task of explaining the rationale of a reasoning as bearing the same learning targets. And, finally, the presentation and management of the tasks in a way that guides students' work but does not prevents creativity is not easy; however, if creativity is not practised, there would be no sense in making a story out of a series of calculations.

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