TEACHERS' BELIEFS ABOUT THE ADOPTION OF NEW TECHNOLOGIES IN THE MATHEMATICS CURRICULUM

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The purpose of the present study was to examine elementary mathematics teachers' concerns in relation to the expected implementation of the new technology based mathematics curriculum. A questionnaire for examining teachers' concerns towards this innovation was administered to seventy four elementary school teachers. Results provide evidence that the majority of teachers were positive towards the innovation. Results revealed the existence of four factors related to teachers' concerns and beliefs towards the innovation, namely the concerns about the nature of the curriculum, teachers' self-efficacy beliefs, concerns about the consequences on the organization of teaching, and concerns about the effectiveness of the curriculum.

INTRODUCTION AND THEORETICAL FRAMEWORK

Based on the premise that Information and Communication Technologies (ICT) can have a positive impact on mathematics teaching and students' learning, a number of countries have implemented in their mathematics curricula technology based activities (Hennessy, Ruthven, & Brindley, 2005). This implementation is, however, not an easy yet straightforward task; a number of factors such as mathematics teachers' beliefs and concerns about the adoption of this innovation, facilities, in-service teachers' training, and available software and resources material might influence the successful implementation of the innovation (Hennessy, et al., 2005).

Gibson (2001) argues that technology by itself will not change schools. It is only when reflective and flexible educators integrate technology into effective learning environments, that the restructuring of the classroom practices will benefit all learners. The introduction and implementation of ICT in the teaching and learning of mathematics has not been successful in a number of cases in different countries (Hennessy, et al., 2005). As reported by the British Educational Communications and Technology Agency (2004), only few teachers succeed in integrating ICT into subject teaching in a fruitful and constructive way that can promote students' conceptual understandings and can stimulate higher-level thinking and reasoning. In most of the cases, teachers just use technology to do what they have always done, although in fact they often claim to have changed their practice. A number of teachers do not feel comfortable with the integration of ICT in subject teaching, since their role was predetermined and designed by educational authorities and teachers feel that they face a lack of professional autonomy (Olson, 2000). Olson (2000) proposes that integrating new technologies challenges teachers and, thus, requires innovators to understand and 'engage in conversations with teachers about their work culture, the technologies that sustain it and the implications of new approaches for those technologies' (p.6).

Among the factors that have been identified as crucial for the successful integration of ICT in the mathematics curricula are teachers' concerns and beliefs about this change (Van den Berg et al., 2000). To this end, a number of studies focused their research efforts on examining teachers' concerns towards the adoption of ICT in general (Gibson, 2001) or towards an innovation in education (Hall & Hord, 2001), since teachers' beliefs and concerns were considered as an important factor of the successful implementation of educational change and reform. Concerns can be described as the feelings, thoughts, and reactions individuals develop in regard to an innovation that is relevant to their job (Hord, Rutherford, Huling-Austin & Hall, 1998). In this framework, innovation concerns refer to a state of mental arousal resulting from the need to cope with new conditions in one's work environment (Hord et al., 1998). Furthermore it is argued that teachers are also important as representatives of their students' needs. In this respect, the opinions and views of teachers can be considered to be reflective of opinions and views from two major stakeholder groups instead of one (Hossain, 2000).

A model that has been widely used for the evaluation of the innovations in education in general is the Concerns-Based Adoption Model (CBAM) (Hord, et. al., 1998). This model can be used to identify how, for example, teachers (who feel that they will be affected by the new technology based curriculum in mathematics) will react to the implementation of the innovation (Christou et al., 2004). The CBAM includes three tools that are used for collecting data related to teachers' concerns and beliefs. These tools include the levels of use questionnaire, the innovation configurations and the stages of concerns questionnaire. The stages of concerns questionnaire was adopted, modified and used in the present study to measure elementary school teachers about the innovation of introducing a technology based mathematics curriculum (Hall & Hord, 2001). The stages of concerns questionnaire includes items for measuring teachers' concerns towards seven stages of concern, namely the Awareness, Informational, Personal, Management, Consequences, Collaboration, and Refocusing stages.

Briefly, in the awareness stage teachers have little knowledge of the innovation and have no interest in taking any action. In the informational stage teachers express concerns regarding the nature of the innovation and the requirements for its implementation. In the personal stage teachers focus on the impact the innovation will have on them, while in the management stage their concerns begin to concentrate on methods for managing the innovation. In the consequences and collaboration stages their concerns focus on student learning and on their collaboration with their colleagues and finally on the refocusing stage teachers evaluate the innovation and make suggestions for improvements (Hord et al., 1998).

PURPOSE AND RESEARCH QUESTIONS

The purpose of the present study was to examine teachers' beliefs about an innovation that will soon take place in Cyprus, namely the adoption of a new mathematics curriculum. The new curriculum is expected to incorporate an inquire based approach and to integrate technological tools into the teaching and learning of mathematics. The study aimed at investigating how well prepared teachers feel about implementing the new curriculum and whether teachers are positive towards this change.

The research questions of the study were the following:

(a) What beliefs do teachers have regarding the adoption of a mathematics curriculum that integrates technology?

(b) Do teachers' beliefs differentiate in accordance to their teaching experience and their studies?

(c) Do teachers feel capable to implement the new curriculum and if not what do they need to be appropriately prepared?

METHODOLOGY

Participants

The participants in this study were 74 teachers from nine elementary schools in Cyprus. Schools were randomly selected from the district of Nicosia. One hundred questionnaires were mailed to schools and 74 were returned to researchers. Teachers were grouped according to their teaching experience and their studies, in three categories and in two categories, respectively. The numbers of teachers in each group are presented in Table 1.

Table1. *Teachers involved in the study by years of teaching experience and level of studies*

Studies	Teaching experience		
	1-5	6-15	>15
Postgraduate studies	16	13	9
Undergraduate studies	6	16	14
Total	22	29	23

Batteries

The questionnaire included 23 likert-scale items. Part of the items was adopted from previous stages of concerns questionnaires (e.g., Hall & Hord, 2001; Christou et al., 2004). Since these studies focused on teachers' adoption of innovations in general, the items were modified to serve the purposes of investigating teachers' concerns of the adoption of the innovation of using ICT in the teaching of mathematics. The 23 items were on a 7-point likert scale, from 1 (strongly disagree) to 7 (strongly agree); all responses were recorded so that higher numbers indicated greater agreement with the statement. The questionnaire also included two open-ended questions in which teachers were asked to report on: (a) what they need in order to feel confident and prepared to implement the new technology-based mathematics curriculum, and (b) their beliefs and concerns in general about their new role in teaching after the implementation of the innovation.

The data were analyzed using the statistical package SPSS. A factor analysis and an analysis of variance were conducted. Descriptive statistics were also used.

RESULTS

The exploratory factor analysis resulted in four factors, including the 21 items of the teachers' questionnaire. The following four factors arose: (a) Concerns/Beliefs about the nature of the new mathematics curriculum, (b) Teachers' self-efficacy beliefs, (c) Concerns about the consequences on the organization of teaching, and (d)

Concerns/Beliefs about the effectiveness of the new curriculum. The loadings of each statement in the four factors are presented in Table 2.

Furthermore, teachers that participated in the study appeared to have positive beliefs about the nature of the proposed new curriculum ($\bar{x}$=5,1). Particularly, the majority of teachers reported that the new curriculum will put emphasis on pupils' way of thinking and their reasoning skills, on problem solving and on the enhancement of students' conceptual understanding. The mean score of the 'Self-efficacy beliefs' factor ($\bar{x}$=4,1) might claim that teachers feel quite confident and well prepared to use the new curriculum. Although the mean score can be considered quite large, is it important to underline that the majority of teachers reported that there is a strong need in-service teachers' training.

Furthermore, it seems that teachers' beliefs concerning the consequences on the organization of teaching are also rather positive. The mean score ($\bar{x}$=4,0) reveals that many teachers who participated in this study believe that after the implementation of the curriculum the stress of the teacher regarding the organization of teaching will be reduced and that this innovation will relieve the teacher from a great deal of

Table 2: *Factor analysis results*

	Factors			
Statements	F1	F2	F3	F4
The adoption of the new curriculum will place sufficient emphasis on the development of pupils' thinking.	,831			
The use of the computer in mathematics develops pupils' mathematical thinking and reasoning skills.	,744			
The new curriculum that takes advantage of the computer in the teaching of mathematics promotes problem solving.	,730			
The use of computers promotes conceptual understanding in mathematics.	,704			
The new curriculum places emphasis on investigation.	,618			
The knowledge that students acquire through the use of computers is not superficial.	,572			
I do not feel confident about teaching mathematics with computers.		,808		
I do not face difficulties in teaching mathematics with computers.		,759		

The implementation of the new curriculum requires the use of methods that I am not familiar with. (recoded)	,723		
I do not need guidance to teach mathematics with the use of computers. (recoded)	,715		
I know how to use computers effectively in mathematics in the classes that I teach.	,541		
The computer based activities that will be included in the new curriculum will reduce teacher's preparation.		,856	
With the implementation of the new curriculum, teachers' stress about the organization of teaching will be reduced.		,846	
Pupils' homework will be reduced.		,578	
Teaching of mathematics with the use of computers will allow me to follow the progress of each pupil.			,775
The adoption of the new curriculum is a useful innovation.			,613
I believe that the adoption of the new curriculum will improve students' achievement.			,557
The integration of computers in mathematics teaching will result in major changes in the teaching of mathematics.			,418

preparation. They also reported that they expect that pupils' homework will be reduced as well and that the integration of technology will improve the organization of the classroom.

Similarly, the mean score for the forth factor was also quite large ($\bar{x}$=5,3). Teachers appeared to be positive that the new curriculum will introduce major changes in the teaching of mathematics and that it will improve results. They also consider the mathematics curriculum that integrates technology as a useful innovation in primary education mathematics and as a means that will allow them to follow the progress of each pupil.

Table 3: *The four factor model mean scores*

Factors	Mean	SD
F1: Beliefs about the nature of the new mathematics curriculum	5,1	0,9
F2: Teachers' self-efficacy beliefs	4,1	1,2
F3: Concerns/Beliefs about the consequences on the organization of teaching	4,0	1,2
F4: Concerns/Beliefs about the effectiveness of the new curriculum	5,3	0,9

In order to investigate whether teachers' beliefs in four factors differentiate in accordance to the years of teaching experience and level of studies, a multivariate analysis of variance was applied with the statements of teachers in four factors as dependent variables and years of teaching experience and studies as independent ones. The results of the multivariate analysis showed that there were significant differences between teachers beliefs across the years of teaching experience (Pillai's $\underline{F}_{(2,64)}$ = 2,211, p<0,05). More concretely, the results indicated that there were statistically significant differences between the three groups only in the first factor, 'Beliefs about the nature of the new mathematics curriculum' (F=5,667, p<0.05). It was found that the significant differences that related to this factor appeared only between inexperienced teachers (years of teaching experience: 1-5) and experienced teachers (6-15) (p<0.05) and between inexperienced teachers and teachers with more than 16 years of experience who probably possess administrative places (16+) (p<0.05). As the years of experience increase the beliefs about the nature of the curriculum get higher. In the rest three factors there were no significant differences between the three groups of teachers. The results of the multivariate analysis indicate that there were no significant differences between teachers' beliefs in the four factors in relation to their level of studies (Pillai's $\underline{F}_{(1,68)}$ = 0,661, p > 0,05).

Of importance are also teachers' responses to a number of individual items of the questionnaire. The item with the highest mean score ($\bar{x}$=6,1) was the one that referred to the need for training courses. Specifically, the majority of teachers (60 teachers), agreed strongly (chose 7) or very much (chose 6), and only two teachers disagreed that training courses are necessary for the successful implementation of the technology based curriculum in mathematics. The items with the lowest mean score were the 'The knowledge that students acquire through the use of computers is superficial' ($\bar{x}$=2,7) and 'The adoption of the new curriculum for the integration of computers in the teaching of mathematics is a useless innovation' ($\bar{x}$=2,1). Teachers' responses to these items also showed that teachers consider the integration of technology in the teaching of mathematics as a useful innovation that will enforce learning, something that is in line with the high mean score ($\bar{x}$=5,2) which refers to the improvement of students' achievement after the implementation of the new curriculum. Their positive beliefs and willingness to integrate technology into teaching appears also from the high mean score ($\bar{x}$=5,2) of the item 'I would like to teach mathematics lessons using computers'.

Table 4: *Mean scores for questionnaire items*

Items	Mean	SD
The knowledge that students acquire through the use of computers is superficial.	2,7	1,2
Training courses for the integration of computers in the teaching of mathematics are necessary for teachers.	6,1	1,3
I would like to observe and participate in technology based mathematics lessons taught by more experienced teachers.	5,2	1,4
I believe that the adoption of the new mathematics curriculum that integrates technology into teaching will improve students' achievement.	5,2	1,1
The adoption of the new curriculum for the integration of computers in the teaching of mathematics is a useless innovation.	2,1	1,7

Teachers' need for training courses came also up from their answers in the first open-ended question. Fifty-five teachers answered this question and some of the answers consisted of a combination of different ideas. For this reason some of the teachers are included in the percentage of more than one category of answers. Forty-six teachers (83,6%) stated that they need 'Training courses for the integration of computers in the teaching of mathematics'. The second category that was pointed out by ten teachers (18%) was 'lesson plans and worksheets'. Also, ten teachers (18%) expressed their need to become familiar with the software that will be used and eight teachers revealed their wish to attend courses that will be held by more experienced teachers. Six teachers stated that they need much guidance, three that they considered the co-operation with colleagues important and three that they need the appropriate infrastructure. The last four answers that were reported only by one teacher each, are the following: (a) training courses for the use of computers, (b) more hours devoted to the teaching of mathematics, (c) one coordinator in each school, and (d) adaptation of the books according to the purpose of the curriculum that integrates technology into teaching.

Regarding the second open-ended question, five categories of answers were identified from the 53 answers that were gathered. The majority of teachers (46 teachers- 88.7%) stated that they feel that their role would be more like a facilitator during the learning process. Three teachers reported that their role will remain the same and two just mentioned that they will have a decisive role. Lastly, one teacher pointed out that his role will change; he will need to first develop more positive attitudes and knowledge towards the innovation and then transfer them to his students.

DISCUSSION

The purpose of this study was to examine teachers' beliefs and concerns regarding the expected innovation of integrating the new technology-based curriculum in mathematics at the elementary schools in Cyprus.

The questionnaire was used to provide a description of teachers' concerns and beliefs about the integration of the new technology-based mathematics curriculum, which shows that the great majority of teachers welcome the expected change in mathematics curriculum after the introduction of ICT and they seem to have positive beliefs in general and positive self-efficacy beliefs for teaching mathematics using ICT (Chamblee & Slough, 2002).

The present study showed that in general teachers welcome the introduction of ICT in mathematics education. According to the teachers that participated in the study, however, the majority of the teachers underlined the importance of in-service and pre-service training on implementing ICT in the mathematics teaching. The results of the study revealed that teachers believe that this innovation is important and can positively change the way mathematics are taught and student learning can be improved, but this is not an easy task; careful planning is needed and resources like software and lesson plans will help teachers in their new different role (Luehmann, 2002).

The results revealed that differences of beliefs across different groups of teachers in terms of teaching experience existed only for the first factor, namely the 'Beliefs about the nature of the new mathematics curriculum'. Specifically, teachers' beliefs about the nature of the curriculum differed between the inexperienced teachers and teachers with more than five years of experience. As teachers' experience increases, teachers feel that the new curriculum can place sufficient emphasis on the development of pupils' thinking and that the appropriate use of computers can assist students in further developing their mathematical thinking and reasoning skills. These teachers also reported that the integration of ICT in the teaching and learning of mathematics can assist teachers in teaching problem solving skills, an essential and core part of the mathematics curriculum.

The themes emerging from the analysis of teachers' beliefs and concerns about the expected integration of ICT in the mathematics curricula converge to offer a grounded model for the innovation. This model underlines the importance of teachers' training and knowledge on the various aspects that are related with the integration of ICT in mathematics. Furthermore, teachers appeared to be very positive about the innovation and that they expect that the role of ICT will assist the teaching and learning of mathematics. This result is very prominent and encouraging, considering that the majority of these teachers were not well informed about the innovation from educational authorities, but were rather themselves positive and believe that the role of technology can positively influence the role of school mathematics.

In the future, a longitudinal study could be conducted to examine the development of teachers' beliefs and concerns over the first steps of the innovation. Since teachers appear to have quite strong and positive beliefs and they expressed their willingness to adopt and use the new curriculum, a study on the development of their concerns and beliefs over a long period could provide more useful information for practitioners and researchers. To better examine the research questions that guided the present study, it is recommended that a comparative study could be conducted to examine the differences between pre-service and in-service teachers' concerns and beliefs towards the new technology based mathematics curriculum, and to identify how the more technology experiences pre-service teachers have might influence their concerns and beliefs about the innovation.

Teachers' beliefs and concerns are an important issue for the successful integration of the ICT in the mathematics curricula, and this study examined this issue in relation to elementary school teachers in Cyprus. it is expected that such explorations can suggest good practices for educational authorities and teacher educators. Finally, the findings discussed would provide avenue and references for future studies.

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DYNAMICAL EXPLORATION OF TWO-VARIABLE FUNCTIONS USING VIRTUAL REALITY[1]

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We present the rationale of an ongoing project, aimed at the development of a Virtual Reality assistant learning of limits, continuity, and other properties in multivariable Calculus. The Mathematics for which this development is intended is described briefly, together with the psychological and pedagogical elements of the project. What is Virtual Reality is explained and details are given about its application to the specific field. We emphasize the fact that this new technological device is suitable for self-teaching and individual practice, as well as for the better storing and retrieving of the acquired knowledge, and for identifying its traces whenever it is relevant for further advanced learning.

BACKGROUND

The institution and its pedagogical situation

The Jerusalem College of Technology (JCT) is a High-Tech Engineering School. During the Spring Term of first year, a course in Advanced Calculus is given, mostly devoted to functions of two, three or more real variables. A problem for many students is a low ability to "see" in three-dimensional space, with negative consequences on their conceptualization of notions such as limits, continuity, differentiability. Another bias appears with double and triple integrals, as a good perception of the integration domain is necessary to decide how to use the classical techniques of integration. Sik-Lányi et al. (2003) claim that space perception is not a congenital faculty of human being. They built a Virtual Reality environment for improving space perception among 15-16 years old students. With the same concern we address a particular problem of space perception with older students, using the same digital technology.

Berry and Nyman (2003) show students' problems when switching between symbolic representation and graphical representation of a 1-variable function and of its first derivative. They say that "with the availability of technology (graphical calculators, data logging equipment, computer algebra systems), there is the opportunity to free the student from the drudgery of algebraic manipulation and calculation by supporting the learning of fundamental ideas". Tall (1991) notes that the computer "is able to accept input in a variety of ways, and translate it's flexibly into other modes of representation, including verbal, symbolic, iconic, numerical, procedural. It therefore gives mathematical education the opportunity to adjust the balance between various

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modes of communication and thought that have previously been biased toward the symbolic and the sequential".

Until now, various technologies have been introduced as a tentative remedy to problems encountered with three-dimensional perception. Nevertheless, problems still remain. Numerous technologies have been introduced for the sake of visualization. Arcavi (2003) classifies the roles of visualization as a) support and illustration of essential symbolic results, b) provider of a possible way of resolving conflicts between (correct) symbolic solutions and (incorrect) intuitions, and c) a help to re-engage with and recover conceptual underpinnings which may easily be bypassed by formal solutions.

In the present paper, we focus on functions of two real variables, plotting and analyzing their graphs, considering especially the b) component in Arcavi's classification. A problem may appear inherent to all kinds of support: a graphical representation may be incorrect, either because of non appropriate choices of the user or because of the constraints of the technology (Dana-Picard et al. 2007). In order to overcome this problem we turn our attention towards another technology: Virtual Reality (VR). This technology is extensively used for training pilots or other professionals. Jang et al. (2007) discuss the usage of VR related to representation of anatomy, clearly a 3D situation too. But as far as the authors know, it has been implemented yet neither for Mathematics Education in general, nor for the Mathematics Education of Engineers. In this paper, we present the rationale for the authors to start the development of a VR assistant to learning Mathematics. We describe an environment where the learner is not passive and has some freedom to choose his/her actions. A VR environment offers cognitive assessment, spatial abilities, executive and dynamical functions which are not present in more traditional environments.

Representations of a mathematical object

Among the characters articulated in mathematics teaching cognitive aspects:

- Multiple representations of the same objects: textual (i.e. narrative) presentations, literal formulas, graphical representations, tables of numerical values, etc. These presentations may either be redundant or leave empty holes. Note that every presentation has to be accompanied by a narrative presentation for embodying the rule and for the sake of completing the given description of a rule. Mathematics educators generally agree that multiple representations are important for the understanding of the mathematical meaning of a given notion (Sierpinska 1992).
- When using together multiple representations in order to give a concrete appearance of composite consequences of the rule under consideration, it can be necessary to perform a transfer between an abstract concept and concrete representations. For example, Gagatsis et al. (2004) present a hierarchy among the possible representations of a function, calling tables as a prototype for

enabling students to handle symbolic forms, and graphical representations as a prototype for understanding the tabular and verbal forms of functions (for a study of prototypes, see Schwarz and Hershkowitz 1999).

- The more numerous the rule's implications (in Physics, Biology, Engineering, Finance, etc.), the more important is the requirement of creative skills (e.g. interpolations, extrapolations, which the learner will have to apply). Here the teacher will generally try and guide the learner with examples, graphical representations, and animations.
- The more fundamental the rule, the more important for the learner to store it, to internalize it and its consequences for a long duration. This will enable him/her to build more advanced rules. More than that, the learner needs ways to extract the knowledge and to find its traces whenever it is relevant for further learning (Barnett et al, 2005).
- Regarding a mathematical rule with geometrical implications and representations, its complete mastering requires from the learner, according to the Gestalt conception, a permanent transfer from one kind of representation to another kind (see Hartmann and Poffenberger, 2007). On the one hand, it is necessary to understand how a change in the parameters of the rule influences the representation. On the other hand, abstraction skills enable to conjecture the rule from the graphical representation and to modify the parameters in the formula according to the changes in the graphical representation. This is the rationale for the usage of software for dynamical geometry.

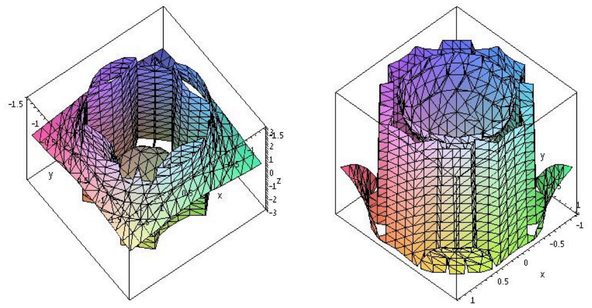
The graphical representation has been made using either Maple 9.5 or the free downloadable software DPgraph (www.dpgraph.com). Because of the dynamic character of a VR device, we do not include screenshots. Suitable presentations can be found at URL: http://ndp.jct.ac.il/companion_files/VR/home.html.

LIMITATIONS AND CONSTRAINTS ON THE CONVENTIONAL REPRESENTATION TOOLS

Real functions of two real variables may have various representations: symbolic (with an explicit analytic expression $f(x,y)= \ldots$), graphical (the graph of the function, i.e. a surface in 3D-space), numerical (a table of values), not necessary all of them at the same time. This last kind of representation is generally not easy to use in classroom; the plot command of a CAS uses an algorithm which provides numerical data, and the command translates this numerical data into a graphical representation. Generally the higher level command is used, and the user does not ask for a display of the numerical output. The VR device that we develop uses this numerical output to create a *terrain* (a landscape) over which the student will "fly" to discover the specific properties of the function, either isolated or non-isolated singularities, asymptotic behaviour, etc.

It happens that a symbolic expression is unaffordable. This creates a need, central for teaching, for suitable tools to illustrate the function and make it more concrete. An example is given by Maple's **deplot** command for plotting the solution of a Differential Equation without having computed an analytic solution; of course this command uses numerical methods. Within this frame, educators meet frequently obstacles for their students to achieve a profound and complete understanding of the behaviour of such functions. Examples of the limitations have been studied by Kidron and Dana-Picard (2006), Dana-Picard et al. (2007) and others. The student's understanding of the behaviour of a given function depends on the representations which have been employed.

Dana-Picard et al (2008) show that the choice of coordinates has a great influence on the quality of the plot produced by a Computer Algebra System (CAS). Compare the plots of $f(x,y)=1/(x^2+y^2-1)$, displayed in Figures 1 and 2. Cartesian coordinates have been used for Figure 1 and polar coordinates for Figure 2. The discontinuity at every point of the unit circle is either not apparent or exaggerated. Moreover Figure 1b shows a kind of waves which should not be there.

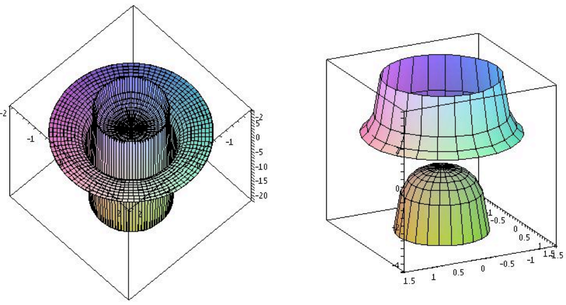


(a) **(b)**

Figure 1: Plots of a 2-variable function, with Cartesian coordinates

The choice of suitable coordinates is not the sole problem for getting a correct plot. Figure 2a shows that our discussion on "correct coordinates" is not the ultimate issue, and even with these coordinates, other choices influence the accuracy of the graph, whence the student's understanding of the situation. In Figure 2a the discontinuities are totally hidden, as a result of the interpolation grid chosen by the software. This issue is discussed by Zeitoun et al. (2008).

A "wrong" choice of coordinates may *hide* important properties of the function, but may *show* irrelevant problems, whence numerous problems with the figure and its adequacy to the study. A central issue is to decide what "correct coordinates" are and what a "wrong choice" is. It has also an influence on the possible symbolic proof of the properties of the function. A couple of students have been asked why they have hard time with such problems; they answered that the reason is a lack of basic understanding of the behaviour of the represented mathematical object (no matter whether the representation is symbolic, numerical, or graphical). A problem can arise when checking that data of two different kinds actually represent the same function. Experience must be accumulated by the learners.



(a) **(b)**

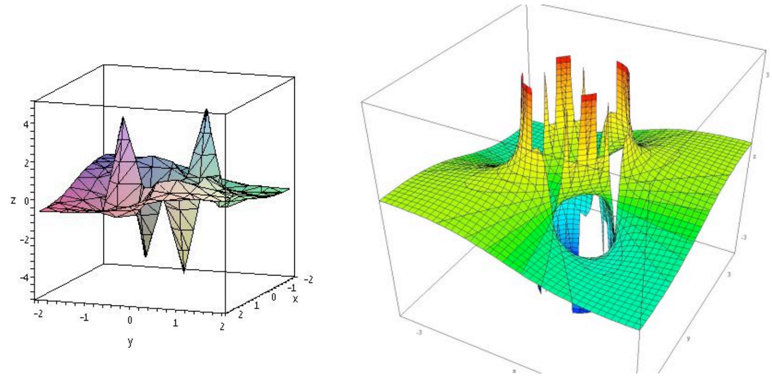
Figure 2: Plots of a 2-variable function, with polar coordinates

Moreover, the students may receive a proof of a certain property using an abstract-symbolic representation of the mathematical object under study. Despite the proof's precision, it happens that the student needs a more concrete presentation. In a practice group of 25 students, the teacher chose the function defined by $f(x,y)=1/(x^2+y^2-1)$ and showed plots like those displayed in Figure 1. Two thirds of the students saw immediately that the function has a lot of discontinuities (intuitively, without giving a proof), but could not explain immediately what is wrong with Figure 1.

The graph of a 2-real variable function is a surface in 3-dimensional space. A function of three real variables can be represented by level surfaces. Excepted at certain points, this is the same mathematical situation as before, because of the Implicit Function Theorem. At the beginning of the course, about 70% of our students have problems with surface drawing. A lack of intuition follows, for example concerning the existence of discontinuities. This may incite the student to make successive trials, i.e. to multiply *technical* tasks not always relying on real mathematical *thinking*. Afterwards a symbolic proof is required, and maybe a graphical representation will be needed to give the "final accord".

Graphical features of a Computer Algebra System are used to enhance visual skills of our students, hopefully their manual drawing skills. With higher CAS skills, an animation of level surfaces can help to visualize graphically a 3-variable function. We meet two obstacles:

- The dynamical features of a CAS are somehow limited. In many occurrences, it is possible to program animations, and/or to rotate the plot, but not more.
- A CAS cannot plot the graph of a function in a neighbourhood of a singular point. In this paper we focus on limits and discontinuities. The CAS either does not plot anything near the problematic point (Figure3b) or plots something not so close to the real mathematical situation (Figure 3b: where do these needles come from?). Note that this occurs already with 1-variable functions, but with 2-variable functions the problem is more striking.



(a) false incomplete plot **(b) incomplete plot**

Figure 3: Two Problematic plots for $f(x,y)=\frac{xy}{x^2+y^2-1}$ **.**

VIRTUAL REALITY

What's that?

The technology called **Virtual Reality** (VR) is a computer-based physical synthetic environment. It provides the user with an illusion of being inside an environment different from the one he/she is actually. This technology enables the building of a model of a "computerized real world" together with interactive motion inside this world. The VR technology gives the user a feeling that he/she an integral "part of the picture", yielding him/her *Presence*, *Orientation*, and even *Immersion* into the scenario he/she is exposed. After a short time he behaves like it's the real world.

The goals: VR-concretization and its added value

A CAS is not a cure-all for the lack of mathematical understanding when dealing with discontinuities of multi-variable functions. A more advanced, more dynamical concretization is given by a VR environment. It is an additional support to Mathematics teaching completing the classical computerized environments, beyond the traditional representations (symbolic, tabular-numerical, and graphical). Actually VR provides an integration of computer modes previously separate (Tall 1991):

- Input is not limited to sequential entry of data using a keyboard. Devices such as a joystick are also used.
- A working session and its output mix together the iconic, the graphical and the procedural modes.

When reacting to the student's commands, the VR device computes anew all the parameters of a new view of the situation. The student takes a walk in a landscape which is actually part of the graph of the function he/she studies. At any time, VR simulates only part of the graph, the discontinuity is never reached, but it is possible to get arbitrarily close to it. The VR may provide the student what is missing in his/her 3-dimensional puzzle, by eliminating the white areas appearing in CAS plots, such as Figure 3a. It is intended to provide him/her a real picture of how the function he/she studies behaves.

A VR environment provides compensation to the limitations and the constraints of the imaging devices already in use (CAS and plotters). It presents an image of a real world and gives a direct 3-dimensional perception of this world, as if the user was really located in it. The higher the quality of the VR environment, the more powerful the impression received from this imaginary world's imitation of the real world.

In our starting project, the simulation provided by VR is intended to improve the students' understanding of continuity and discontinuity, and afterwards give also a better understanding of differentiability of a multi-variable function. Among other affordances, the VR simulation cancels problems of discontinuity related to graphs because of its local and dynamical features.

COGNITIVE CHARACTERISTICS AND SIMULATION FEATURES OF A VR ENVIRONMENT

The final rules may be represented in a concrete fashion by interaction with the environment and by showing to the learner the limitations of the rules, as they appear in a (almost) static environment generally yielded by a CAS. Non graphical representations of functions, such as numerical representations, cannot show continuity and discontinuity. This comes from the discrete nature of these representations, a feature still present in the computerized plots.

The new knowledge afforded by the learner is a consequence of his/her own efforts to explore the situation. His/her ability to change location, to have a walk on the graph, will lead him/her to internalize in a better way the mathematical meaning of continuity and discontinuity. An added value is to help him/her to understand the meaning of changing parameters in the geometric representation. This added value is made possible by the *live experience* of the behaviour of the function, no matter if the transitions are discrete or continuous (according to changes in the variables or in the parameters). The mental ability to feel changes, their sharpness, their acuteness, comes from the immersion into the topography in which the learner moves.

This added value is still more important when the function under study encodes a concrete situation, in Physics, Engineering, Finance, etc. The interactive experience enables the learner to translate the rules to which the function obeys, to find analogues of these rules for other concrete situations. The concrete sensations provided by VR improve the learner's understanding of interpolation and extrapolation, and to translate this understanding into the graphical situation (see also Dana-Picard et al., 2007). The more immersive features of the mathematical knowledge that are incorporated into VR representation for the learner, the faster he/she will find the traces of it whenever it is relevant for further learning. Besides, the more immersive features are incorporated into VR knowledge representation the greater the longevity of preserving the acquired knowledge. This means a slower extinction of it in the memory system (Chen, et.al. 2002).

Interactivity improves the learning experience. Numerous studies show that the more deeply lively experienced the learning process the more internalized its results (Ausburn and Ausburn, 2004; Barnett et al, 2005). The internalization is assessed by an improved conservation of the knowledge, i.e. a slower decrease of the knowledge as a function of elapsed time. Therefore, a Virtual Reality assisted learning process yields a better assimilation of the mathematical notions than with more conventional simulations devices, as it provides this live sensorial experience. This is a more than a realization of the request expressed by a student involved in a research made by Habre (2001); this student wished to be able to rotate surfaces in different directions. A Computer Algebra Systems does this already. VR meets a further requirement of this student, namely to have "a physical model that you can feel in your hands".

According to the brain mapping, the numerical representation of functions is acquired by the left hemisphere of the brain, and the space-live experienced acquisition in a learning process is devoted to the right hemisphere. The transfer from the symbolic rule to a 3D representation and vice-versa requires transfer between two brain lobes with different functionalities. Concerning conceptualization, especially when it must be applied to a concrete domain, there exists a mental difficulty to "move" from one lobe to the other (in terms of longer reacting time, or of completeness of the process). An interactive environment where functional parameter changes are allowed, and where the environment changes can be sensitively experienced, enables a faster building of bridges between the different registers of representation, symbolic, numerical, and graphical.

Finally, the usage of a VR assistant to learning is purely individual. The teacher can show a movie, but it is only an approximation of the requested simulation. The student's senses are involved in the process, the hand on the joystick, the eyes and the ears in the helmet, etc. Therefore the VR device should take in the learning computerized environment a place different from the place of other instruments.

OUR VR DEVICE AND FUTURE RESEARCH

The digital device described above is now in its final steps of initial development. The user can fly over (or walk along) the terrain, i.e. over the graph of the given function. The details of the graphs, the possible discontinuities, are made more and more visible. This effect is not obtained by regular zooming, as this operation only inflates the size of the cells of the interpolation grid. For new details to appear the data has to be computed anew and only part of the surroundings is displayed.

Furthermore, a VR environment seems to contribute an added value by representing more holistic characteristics of the mathematical knowledge. Among the main contributions are the dynamics or flow traits. A more integrated one is the ability to understand its place in the whole mathematical or physical context it is playing with.

In cognitive terms it means that by VR environment, the teacher should provide to the student a more accurate mental model of the mathematical knowledge, including the applicable images of it (Croasdell et al, 2003).

In particular, the dynamical properties of a VR device and their appeal to various sensitive perceptions (vision, audition, etc.) induce also the need of the integration of the hand into the educative schemes. As Eisenberg (2002) says, the hand is not a peripheral device, but is as important as the brain. He discusses the issue of the importance of physical approximations to purely abstract concepts, rejected by Plato's point of view. Here we use the hand totally coordinated with vision and sensorial perception.

As noted by Artigue (2007), "The increasing interest for the affordances of digital technologies in terms of representations have gone along with the increasing sensitivity paid to the semiotic dimension of mathematical knowledge in mathematics education and to the correlative importance given to the analysis of semiotic mediations". In this perspective, a preliminary double blind research is on its way, with two groups of JCT students. We intend to report on the results in a subsequent paper.

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HOW DIGITAL ARTEFACTS CAN ENHANCE MATHEMATICAL ANALYSIS TEACHING AND LEARNING

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Digital technologies seem to be still very promising to fruitfully support the construction of mathematical knowledge. Even more interesting is the way to incorporate them into the design of a learning environment framed by certain institutional constraints. Through this study we present some reflections and ideas arising from the dialectic interplay between the environment and the students in their effort to formulate a calculus theorem and construct its proof. Related teaching and learning phenomena providing information on instrumental genesis processes are primarily discussed.

INTRODUCTION

Elementary pre-calculus is at the heart of the syllabus at secondary level mathematics education and the entry-point to undergraduate mathematics as well. Many research studies witnessing students' problems to attain a satisfactory level of conceptualisation have been held on this field (for example, see Artigue, 1999). This fact is related to mathematically superficial strategies (Lithner, 2004) implemented by traditional procedure-oriented teaching practices which are generated by both teachers' attitudes and institutional constraints implicitly or explicitly imposed by textbooks and curricular objectives (Ferrini-Mundy & Graham, 1991). Even at the university level, this situation results in detecting serious difficulties on behalf of the students when faced with non-algorithmic type demands which entail reasoning and conceptual understanding (Gonzales-Martin & Camacho, 2004).

On the other hand, the development of mathematics has always been dependent upon the material and symbolic tools available for mathematical computations (Artigue, 2002). Current research on mathematics education regarding the relationships between curriculum, classroom practices and software applications (Lagrange, 2005) offers the ground to address and develop questions concerning technology's fitting into learners' actual social and material environments, the problems users have that technology can remedy, and, furthermore, ways of conceptualizing the design of innovative learning tools as emergent from dialectics between designers and learners-users of those tools.

The learning environment is supported by a Dynamic Geometry software (DGS) enhanced by a function-graphing editor to help Mathematical Analysis teaching at the level of 12th grade.

The whole didactic sequence produced covers the introduction of global and local extrema definitions, Fermat theorem with its proof, the mean value theorem, monotonicity definitions and the derivative sign/function variation theorem along with the proof and its applications. Selection of the exact targeted mathematical material on the field of differential calculus , as well as further elaboration of the activities were attempted with the intention to form a rational succession of concepts to a coherent local unity of mathematical knowledge, including introduction of definitions, formulation of theorems and construction of proofs. From this still on-going research, we present here some elements derived only from an activity concerning the teaching and learning of Fermat theorem's formulation and proof.

THEORETICAL FRAMEWORK

Complexity and close interweaving of cognitive, institutional, operational and instrumental aspects obliged us to adopt a multidimensional approach (Lagrange et al, 2003) in order to design the learning environment and study the teaching/learning phenomena produced.

According to Duval (2002), construction of mathematical knowledge is strongly attached to the manipulation of different semiotic representations. This term refers to productions made up of the use of signs and formed within a semiotic register which has its own constraints of meaning and function. More specifically he defines a "register of semiotic representation" as a system of representations by signs that allows the three fundamental activities tied to the processes of using signs: the formation of a representation, its treatment in the same register, its conversion to another register. Interaction between different registers is considered to be of great importance and necessity to achieve understanding of a mathematical concept. Under this aspect, our tools were designed with the intention to mobilise and flexibly articulate semiotic representations within the numerical, the algebraic and the graphical register and, finally, to generate mathematical conjectures.

Very special and idiomorphic conditions existing within the local educational culture of Greek 12^{th} grade students obliged us to take into consideration the notion of didactical transposition (Chevallard, 1991). At this level a huge amount of institutional pressure results in the development of an "exam-oriented mentality" on behalf of the students as well as their families, which promotes a procedure-oriented attitude towards the mathematical knowledge in context. Candidates' needs to be prepared for a final national examination to enter the university at the end of the year results, finally, in an implicit (or even sometimes explicit!) meta-didactical attitude leading them to ignore or reject conceptual approaches not strongly attached to exam demands. Through this perspective we were obliged to take into account and reinforce the epistemic value of the mathematical knowledge to be taught without any decrease or discount of the pragmatic one (Artigue, 2002), in the economy of the available didactical time. Relating this idea to the tools' design, we considered the possibility to teach basic mathematical concepts within a reasonable amount of learning time, and in ways compatible to both its institutional dimension and the transition to advanced mathematical thinking.

The theory of didactic situations (Brousseau, 1998) helped us conceive the whole learning environment (milieu) as a source of contradictions, difficulties, and disequilibria stimulating the subject (through an effort to control it) to learn by means of adaptations to this environment. At this point we took also into account activity theory (originated in socio-cultural approaches and mediation theories rooted in Vygotski, 1934) to assign to the environment a character sometimes antagonistic to the subject (as pointed by TDS) but also sometimes cooperative and oriented to an educational aim, guided by distinctive didactical intentions.

In order to best incorporate digital artefacts in our didactical engineering we considered the potential technology offers for linking semiotic registers within the frame introduced by the instrumental approach (Rabardel, 1995, Artigue, 2002, Trouche, 2004). A cultural tool or artefact designed to mediate mathematical activity and communication within a socio-cultural context differs from the corresponding instrument into which this artefact can be transformed. The final result is a construction by the subject, in a community of practice, on the basis of the given artifact by means of social schemes. This transformation is developed through an instrumentation process directed towards and shaping the subject's conceptual work within the constraints of the artifact and an instrumentalisation process directed towards and shaping the artifact itself. Both constitute a bidirectional dialectic and sometimes unexpectedly complex process called instrumental genesis (Artigue, 2002). Concerning tool design, we tried to keep simplicity and friendliness to the user, in the sense that their implementation demands, as far as possible, a short process of appropriation by the user and an easy way to be transformed into mathematical instruments to be utilised in the context of the activities. The necessity of any technical support by the teacher was also minimised as far as possible.

The crucial question to answer through our research is whether a design philosophy under the norms mentioned above has the potential to determine a set of effective digital learning tools pre-constructed on the dynamic software which can be easily transformed to learning instruments successfully integrated into the teaching of important calculus concepts at the level of theorem formulating and proof. By the term *successfully integrated* we mean that, firstly, they can make visible phenomena previously invisible, secondly, they can potentially generate innovative approaches to important mathematical concepts, and, thirdly, they lead to a better understanding of any productive or problematic dimension of the computer transposition of mathematics knowledge achieved by the instrument.

METHODOLOGY

The activity (of total duration 90 min) was developed in two different schools in groups of 12th grade students (12 in one group and 9 in the other) during the month of February, 2008. The main differences between the students of the different schools were identified more on the socio-cultural and financial background of the corresponding families (we did not address any comparison issue in our research goals) and less to the fact that comparatively more students belonging to a certain school possessed a certain familiarity with mathematics software use, being exposed several times in the past at different kinds of technology enhanced approaches. For the latter we did not find enough evidence to support the idea that different software cultures established by the students have great impact on their attitude and capabilities of manipulating the pre-constructed software tools induced by our activities.

The informatics laboratory of every school was used and the pupils were at couples situated in a PC-environment. This time the researcher played the role of the teacher as an orchestrator of the in-class situations. A Teacher-Analysis sheet has also been developed to provide necessary details so that other teachers can handle the in-class orchestration.

At the beginning, a worksheet was given to the students to work with and at the end of the session they received a corresponding post-assessment sheet including questions of mathematical nature, which they returned back completed after one or two days. The whole didactic sequence (consisting of four Sessions) was recorded by a voice-recorder and since the whole sequence being completed, a post-questionnaire was passed to the students in order to collect and save some of the prints their instrumental approach had left on them. Finally, four students (two for each group) were interviewed to explicitly clarify their answers at this questionnaire concerning the instrumented actions performed and the students' attitude towards mathematics teaching before and after the whole experience.

The way of obtaining results serving the a posteriori analysis from the raw input data has to be explained here. The whole content referring to the 2nd Activity (Fermat Theorem) has been divided (according to the conceptual meaning development) into 12 *Episodes* and each one of them potentially to one up to four *Phases*. Next, for every one of the 24 *Phases* produced, we used the transcribed outcomes of the recorded class discourse, along with the written notes and answers of the students on the worksheet to produce some discrete entities of information we called *Events*. An *Event* in this terminology is characterised and differentiated by components of mathematical or didactical or instrumental nature which can probably coexist. The study and analysis of these *Events* provided our a posteriori analysis with the material to compare the results composed up to this point with the analysis of the students' answers to the corresponding post-assessment sheet being sorted and analysed separately. Finally, we took into consideration the students' answers on the final post-questionnaire as well as the transcribed explicitation interviews to enhance our vision and come up to some final conclusions.

LEARNING ENVIRONMENT

Concerning the tools' design (and being sensitive to the complexity of instrumental genesis processes), we tried to reduce, at least, the complexity of the interface. We tried, as well, to keep tools' implementation strongly attached to the emerging mathematical needs. The learning environment regarding the whole activity was, thus, perceived with the intentions to:

a) Mobilise students' interest to estimate local extrema departing from a real problem b) Make up a link with the students' previous knowledge on the subject of local extrema and the limit concept c) Stimulate the students to construct the targeted mathematical knowledge by mobilising different registers of representation (graphic, numerical, symbolic, and verbal) for the same concept and favouring representational interconnections between them d) Use the in-class discourse to generate an activity space favouring students' effective instrumental processes e) Support conjecturing, conceptualisation, and institutionalisation f) Insert certain examples and counter-examples when necessary (Gonzales-Martin and Camacho, 2004).

We focus on one of the activities designed to introduce the concept of Fermat theorem. Our specific didactical aims were for the students: to conjecture Fermat theorem, construct its formal statement and proof realising the absolute necessity of its presuppositions and its application range, to perceive that the opposite form of the theorem is not valid, and, finally, to apply the theorem in calculating the local extrema of the function within the problem when its formula is given.

In the following we describe and analyse some selected *Events* drawn out of two different *Episodes*. The material that will be presented is coming from a blend of actual events produced by both groups of students, whose comments and actions have been complementing each other over the flow of the activity.

Remark: The term S-Tools refers to the specific on-Screen pre-constructed tools on the software.

Episode A: Introduction to the *Line y*=k, *IntersectionPoints*, and *Magnification* S-Tools and their application in approximating local extrema positions on the function graph.

Tool Description: The students were prompted to open *Line y=k* and *Intersection Points* S-Tools. The first one draws a horizontal parametric line, whose position can be controlled by the active parameter k (a number in yellow background on the screen that can be modified by the user, see Image 1). If this line has some common points with the function graph then the second S-Tool *IntersectionPoints* draws these intersection points and provides their x-coordinates. Furthermore, a technique permitting the students to change the decimal length and the digits of any active parameter was explained to them by the teacher.
The following question given by the corresponding worksheet came to stimulate students to S-Tools utilisation:

Q1: Could you find or estimate points of local extrema for function P?

Aim Description: The main intention of the constructed situation was to encourage students to explore and use the S-Tools in order to estimate several intervals of x-axe that could enclose positions of internal local extrema and to get approximate values for these positions by shortening the length of the corresponding intervals. Moreover, they had to identify the kind of local extremum (maximum or minimum) and perceive which of them are internal to the interval.

Events: The teacher asked the students to change the active parameter k and see what happens. Some of them could not understand the changes on the counters of intersection points coordinates and that was clarified by the related discussion in the class community. Then, the students were asked to use these tools to numerically estimate the local extrema positions (Question Q1) on the graph the better they could do (Image 1). Some students could not cope with changing the decimal length and the values of several digits so they were given additional technical instruction for that. The teacher asked them to find an interval including the abscissa of a local maximum (this was done very easily) and then to try to shorten this interval by means of the tool. This was not so easily done by every pupil but remarks made by several students and on-screen indications gave good results.

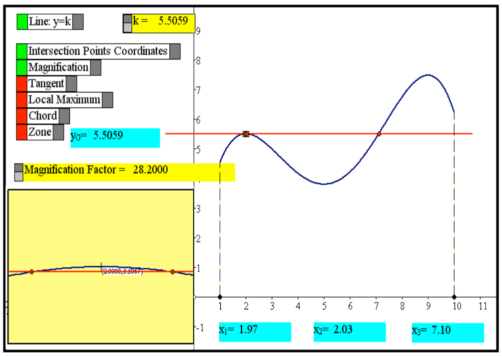

Image 1

Interesting events identified on behalf of the students were:

-Six of them noticed that they could see intersection points on the screen but the indications on the coordinate counters did not attest such an existence.

-During exploring with decimal digits 8 of them observed two intersection points approaching each other and, finally, coinciding to only one but the indications on the corresponding counters were different.

Concerning these two events, the teacher's proposition was to use the *Magnification* S-Tool.

Tool Description: This S-Tool could be used to magnify a selected region around a moving point on the graph and it is controlled by the *Point-Abscissa* and the *Magnification Factor*.

Subsequently, the students were asked to use the same process to estimate the values of every local extremum they could perceive on the graph.

Remarks: Students' written answers on the worksheet revealed that the whole class succeeded at the qualitative level (number of local extrema, approximate position and characterisation). However, at the numerical level, only a small part of them tried to test at the most the instrument's potentialities (and even less achieved at exhausting them) providing the values asked at 3rd or 4th decimal digit accuracy as we had anticipated. Technical weaknesses versus time disposal and partial disinterest have been estimated as some of the possible reasons for that.

Results: This first contact with the notion of approximation opened up the ground for a further in-class discussion. The discourse came up to the point that the tool is able to provide visual images of a certain validity only as an indication generator (which in certain cases can be of great importance for the mathematical knowledge) but not always to produce an arithmetic value in absolute accuracy. The teacher reinforced this situation by asking what would happen if the extremum in search had the real value $2/3$ or $\sqrt{2}$. This fact conducted the discussion to bring into light the inherent inadequacy of every computing system to represent infinite decimal numbers in a complete way. So the students realised that, through this attempt, and also in general, they could obtain only relative accuracy for the local extrema values. The necessity of devising new mathematical tools that could probably provide absolute accuracy for these values came in the discourse.

Episode B: Introduction to the tangent: Relating line y=k when passing through an internal local extremum to the function graph – Derivability

Next Question Q2 had the intention to sensitise students' attention and make them focus to what is going on locally at the area of an internal local extremum point.

Q2: When line *y=k* is passing through an internal local extremum point on the graph, how is this line related to the curve at an area near this point?

Description: Within this *Episode* the students were asked to express their thoughts regarding the visual relation between the line y=k when passing through an internal local extremum point on the graph and the curve itself near the extremum point. The first attempt was made on normal view and the second by means of the *Magnification* S-Tool (Image 2). Subsequently, at the third phase of the *Episode* a new subroutine program file was invoked, where the students could alternatively observe under magnification the behaviour of functions $y = x^2$ and $y = abs(x)$ in the neighbourhood of $x = 0$ (Image 3). This was done by changing only the function formula through a menu of the file. The technique for that was shortly explained by the teacher.

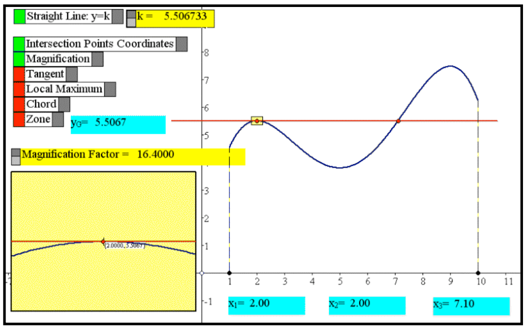

Image 2

Events: The class discourse developed at this *Phase* helped many of the students to communicate their thoughts and formulate them in an intelligible way. They came up with the visualisation of the inequality relations $f(x) \le k$ or $f(x) \ge k$ near the local extremum. Relating this fact to the image produced by the function graph and the horizontal line, they could easily conjecture that this line when passing through a local extremum point on the graph *"leaves the whole curve on one side"* or *"does not cut it"* at the area near this point (having probably in mind a counter-image like a $y = x^3$ graph with the tangent at $x = 0$).

Remarks: Analysis of students' written answers on the worksheet showed that the big majority of them succeeded in perceiving the visual relation between the curve and the line and, moreover, some were able to connect it with the corresponding symbol relation. Four students proceeded to conjecture that, at this case, this horizontal line should be a tangent of the graph, whereas even fewer (two students) mentioned that there was only one common point of the line and the curve at the area near the local extremum.
To the question of the teacher if these two conditions (existing of a single common point and "not cutting" in the area near a local extremum) are adequate enough to assure the existence of a local extremum, confusion arose and the community was not able to provide a clear answer. This event, along with the term tangent mentioned earlier, was used as a bridge to the discussion of next question:

Q3: At the area near the extremum point, can you observe any additional relation between the curve and the line *y=k* when the latter is passing through this point?

Remarks: Class discourse concerning this question resulted in the assertion on behalf of the students that under magnification the curve tends to become a horizontal line or to coincide with it. Moreover, there were some more students stating in a clear way the conjecture that the horizontal line when passing through a local extremum point on the graph *keeps the position of a tangent of the graph at this point*. This conjecture provided the bridge through which the teacher introduced the issue of the existence of the tangent at such a point. Additionally, as a natural consequence of the previous discussion, the subroutine file was used to support students' exploring and help them visualise the difference between the function graphs of $y=x^2$ and $y=abs(x)$ on point $x=0$ under magnification (Image 3) and relate it to the derivability of the function at this point. Most of the students' expressions were for example *"Oh, there's an angle there!"* or *"... in this case we have a peak point ..."* etc.

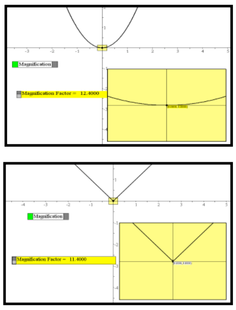

Image 3

DISCUSSION AND PRELIMINARY RESULTS

In this paper, we tried to describe a few situations concerning only the instrumental dimension of our research. The *Episodes* presented above contribute to the first step to Fermat theorem's construction departing from an intuitive approach. This is achieved by exploring and visualizing the local extrema positions and theorem's presuppositions, as well.

As it has been pointed by Guin and Trouche (1999), students' answers were strongly dependent on the environment:

At a first attempt, many students tried to configure the artefact regarding the needs of the specific work: screen view adaptation by transposition of toolboxes and active parameters configuration (i.e. changing the decimal length and the values of certain digits of parameter *k*). These facts confirm, on their behalf, an effort to adapt the artefact to the demands of the specific task induced by the first question Q1

(Estimation of local extrema values) and we consider that as a step to the direction of instrumentalisation in the evolution of instrumental genesis processes (Rabardel, 1995, Trouche, 2004). As instrumentation processes had intently been designed and anticipated not to be very complex, soon after, we could observe automaticity towards certain instrumented action schemes to the execution of necessary tasks (i.e. utilising active parameters).

We point to an internal constraint (Trouche, 2004) of the instrument, which is related to computer's inherent deficiency in providing absolute preciseness through computations, regarding infinite decimal numbers. This issue was discussed with the students during several activities and, finally, was used as an entry to the discussion concerning the notion of approximation. Additionally, a common feeling was developed pointing out that computers will not solve all the mathematical questions inserted. This fact was also used to encourage students to develop their knowledge so as to overcome these limitations.

Students' answering to questions of the post-assessment sheet regarding Fermat theorem's statement or its negation or its applications within only the graphic register showed that the great majority of them (18 out of 21) could cope very good at this level. However more complex questions relating this register to the algebraic one have been more or too difficult for the students, proving that more work is necessary to be done at this level.

Analysis of students' answers to the final post-questionnaire testified a generally positive attitude towards *"this way of teaching"*. For example, to the question: "*Could you identify any positive or negative points through this series of activities you have been attending?*", some of their answers were: *"We could discover and see by ourselves most of the things on the screen…"* or *"By the aid of the computer we could really see and work on the staff we treat usually in the class"*, or *"It was easy-going because we first made the proof of the theorem and at the end we got the typical statement"* or *"It was very helpful to recollect the images on the screen, but the problem was that we didn't solve many exercises!"* etc. Of course, more work and analysis need to be done on this subject in order to obtain some reliable results.

Due to the lack of space, we did not address issues concerning the ways the rest of our theoretical perspectives shape our research. However, some of the first results, elaborated up to the moment, seem to deepen our reflection. They show the potential of such a learning environment design to produce didactical phenomena giving an illumination to both problematic and productive aspects of the mathematical knowledge developed through the educational use of digital technologies.

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INSTRUMENTAL ORCHESTRATION: THEORY AND PRACTICE[2]

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Abstract. *The paper concerns the way teachers use technological tools in their mathematics lessons. The aim is to investigate the explanatory power of the theory of instrumental orchestration through its confrontation with a teaching episode. An instrumental orchestration is defined through a didactical configuration, an exploitation mode and a didactical performance. This model is applied to a teaching episode on the concept of function, using an applet embedded in an electronic learning environment. The results suggest that the instrumental orchestration model is fruitful for analysing teacher behaviour, particularly in combination with additional theoretical perspectives.*

INTRODUCTION

The integration of technological tools into mathematics education is a non-trivial issue. More and more, teachers, educators and researchers are aware of the complexity of the use of ICT, which affects all aspects of education, including the didactical contract, the working formats, the paper-and-pencil skills and the individual and whole-class conceptual development.

A theoretical framework that acknowledges this complexity is the instrumental approach (Artigue, 2002). According to this perspective, the use of a technological tool involves a process of *instrumental genesis*, during which the object or artefact is turned into an instrument. The instrument, then, is the psychological construct of the artefact together with the mental schemes the user develops for specific types of tasks. In such schemes, technical knowledge and domain-specific knowledge (in our case mathematical knowledge) are intertwined. Instrumental genesis, in short, involves the co-emergence of mental schemes and techniques for using the artefact, in which mathematical meanings and understandings are embedded.

Many studies focus on the students' instrumental genesis and its possible benefits for learning (e.g., see Kieran & Drijvers, 2006). However, it was acknowledged that instrumental genesis needs to be guided, monitored and orchestrated by the teacher. In order to describe the management by the teacher of the individual instruments in the collective learning process, Trouche (2004) introduced the metaphorical theory of instrumental orchestration.

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Until today, however, the number of elaborated examples of instrumental orchestrations is limited. Therefore, the aim of this paper is to investigate the explanatory power of the theory of instrumental orchestration through its confrontation with a teaching episode. As such, this contribution can be situated in the intersection of themes 2 and 3 of Cerme6 WG7: it concerns the interaction between resources or artefacts and teachers' professional practice, in which students use tools in their mathematical activity.

In the following, we first define instrumental orchestration. Then a description of a classroom teaching episode in which a technological tool plays an important role is provided. The episode is analysed in terms of the theory. This is followed by a reflection on the application and the conclusions which we have drawn.

INSTRUMENTAL ORCHESTRATION: A THEORETICAL MODEL

The theory of instrumental orchestration is meant to answer the question of how the teacher can fine-tune the students' instruments and compose coherent sets of instruments, thus enhancing both individual and collective instrumental genesis.

An *instrumental orchestration* is defined as the intentional and systematic organisation and use of the various artefacts available in an – in our case computerised – learning environment by the teacher in a given mathematical situation, in order to guide students' instrumental genesis. An instrumental orchestration in our view consists of three elements: a didactic configuration, an exploitation mode and a didactical performance.

1. *A didactical configuration* is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it. These artefacts can be technological tools, but the tasks students work on are important artefacts as well. Task design is seen as part of setting up a didactical configuration.

   In the musical metaphor of orchestration, setting up the didactical configuration can be compared with choosing musical instruments to be included in the orchestra, and arranging them in space so that the different sounds result in the most beautiful harmony.

2. *An exploitation mode* of a didactical configuration is the way the teacher decides to exploit it for the benefit of his didactical intentions. This includes decisions on the way a task is introduced and is worked on, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students.

   In the musical metaphor of orchestration, setting up the exploitation mode can be compared with determining the partition for each of the musical instruments involved, bearing in mind the anticipated harmonies to emerge.

3. *A didactical performance* involves the ad hoc decisions taken while teaching on how to actually perform the enacted teaching in the chosen didactic configuration and exploitation mode: what question to raise now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool?

   In the musical metaphor of orchestration, the didactical performance can be compared with a musical performance, in which the actual inspiration and the interplay between conductor and musicians reveal the feasibility of the intentions and the success of their realization.

The model for instrumental orchestration initially was developed by Trouche (Trouche 2004) and included the first and the second points above, i.e. the didactical configuration and the exploitation mode. As an instrumental orchestration is partially prepared beforehand and partially created 'on the spot' while teaching, we felt the need for a third component reflecting the actual performance. Establishing the didactical configuration has a strong preparatory aspect: often, didactical configurations need to be thought of before the lesson and cannot easily be changed during the teaching. Exploitation modes may be more flexible, whereas didactical performance has a strong ad hoc aspect. Our threefold model thus has an implicit time dimension.

The model also has a structural dimension: an instrumental orchestration on the one hand has a structural, global component in that it is part of the teacher's repertoire of teaching techniques (in the sense of Sensevy 2005) and can be reflected in operational invariants of teacher behaviour. On the other hand, an instrumental orchestration has an incidental, local actualisation appropriate for the specific didactical context and adapted to the target group and the didactical intentions.

The instrumental orchestration model brings about a double-layered view on instrumental genesis. At the first level, instrumental orchestration aims at enhancing the students' instrumental genesis. At the second level, the orchestration is instrumented by artefacts for the teachers, which may not necessarily be the same artefacts as the students use. As such, the teacher himself is also involved in a process of instrumental genesis for accomplishing his teaching tasks (Bueno-Ravel & Gueudet, 2007).

In literature, the number of elaborated examples of instrumental orchestrations is limited. Trouche (2004) and Drijvers & Trouche (2008) describe a so-called Sherpa orchestration. Kieran & Drijvers (2006), without mentioning this orchestration explicitly, describe an instrumental orchestration of short cycles of individual work with the artefact and whole-class discussion of results.

THE CASE OF TWO VERTICALLY ALIGNED POINTS

The case we describe here stems from a research project on an innovative technology-rich learning arrangement for the concept of function[3]. In this project, a learning arrangement for students in grade 8 was developed, aiming at the development of a rich function concept. This includes viewing functions as input-output assignments, as dynamic processes of co-variation and as mathematical objects with different representations. The main technological artefact is an applet called AlgebraArrows embedded in an electronic learning environment (ELO). The applet allows for the construction and use of chains of operations, and options for creating tables, graphs and formulae and for scrolling and tracing. A hypothetical learning trajectory, in which the expected instrumental genesis is sketched, guided the design of the student materials. An accompanying teacher guide contained suggestions for orchestrations.

After group work on diverse problem situations involving dependency and co-variation, the notion of arrow chains is introduced to the students. In the third and fourth lessons, students work with arrow chains in the ELO. One of the tasks of the fourth lesson, which some of the students did at home, is task 8, shown in Figure 1.

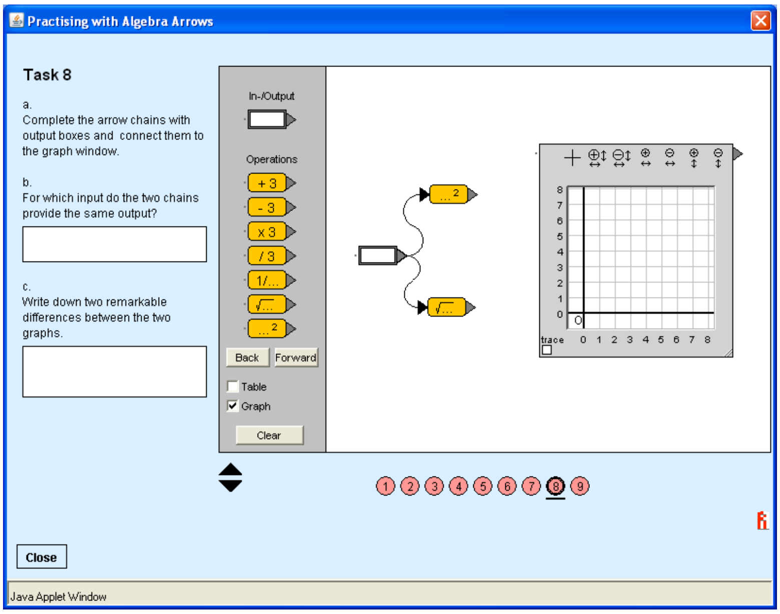

Figure 1 Computer task 8

At the right of Figure 1 is the applet window, which in this task contains the start of the square and the square root chain, and an empty graph window. At the left you see the tasks and two boxes in which the students type their answers. The numbered circles at the bottom allow for navigation through the tasks.

The following verbatim extract describes the way the teacher discusses this task during the fifth lesson.

[3] For further information on the project see Drijvers, Doorman, Boon, Van Gisbergen & Gravemeijer (2007) and the project website www.fi.uu.nl/tooluse/en/.

Using a data projector, the ELO with the list of student pairs is projected on the wall above the blackboard. The teacher T navigates within this list to Tim and Kay's solution for task 8.

T: It says here [*referring to question c*]: what do you notice? Oh yes, I actually wanted to see quite a different one, because they had …

T navigates to Florence and her classmate's work. The Table option is checked. That leads to 'point graphs' on the screen. The students' answer to question c reads:

"For the square they are all whole numbers, and for the square root they are whole numbers and fractions. And the square of a number is always right above the root. ?"

T: Look here, what this says. [*indicates the students' answer of question c on the screen with the mouse*] For the square they are all whole numbers, okay, and for the square root they aren't whole numbers, we agree with that too, and the square of a number is always right above the square root.

F(lorence): Was that right?

T: I'm not saying.

St1[4]: Yes, I had that too.

T: What they say, then, is that every time there is…if I've got something *here*, there is something above it, and if I've got something *there*, there is also something above it. [*points vertically in the graph with the mouse*] Why is that, that these things are right above each other?

F: Well, because it…the square root is just…no the square is just, um, twice the root, or something.

St2: No.

T: Kay?

Kay: That's because the line underneath, that's got a number on it, which you take the square root of and square, so on the same line anyway.

T: What are those numbers called that are on the horizontal line then?

St3: The input numbers.

T: The input numbers.

T: Ehm, Florence, did you follow what Kay said?

F: No, but I [...]. It was about numbers and about square roots and about…

Sts: [*laughs*]

St: It was about numbers!

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[4] St1, St2, .. stands for one of the students

T:	Kay said: these are the input numbers, here on the horizontal line. [*indicates the points on the horizontal axis with the mouse*] And for an input number you get an output number. And that is right above it. So if you take the same input number for two functions… [*indicates the two arrow chains with the mouse*]
F:	Oh yes.
T:	… then you also get…then you get points above it. So that's got nothing at all to do with the functions. It's just got to do with from which number you are going to calculate the output value. Now, if for both of them you calculate what the output value is for 10, they both get a point above the 10 [indicates on the screen with the mouse]. Do you understand that?
F:	Oh yes, I didn't know that.
T navigates back to the list of student pairs.	

Figure 2 shows the work of Florence and her classmate on this task in Dutch at the end of the teaching sequence. They changed their answer to question c into: "for the square they are always whole numbers, and for the square root they are whole numbers and fractions. The squares get higher with much bigger steps."

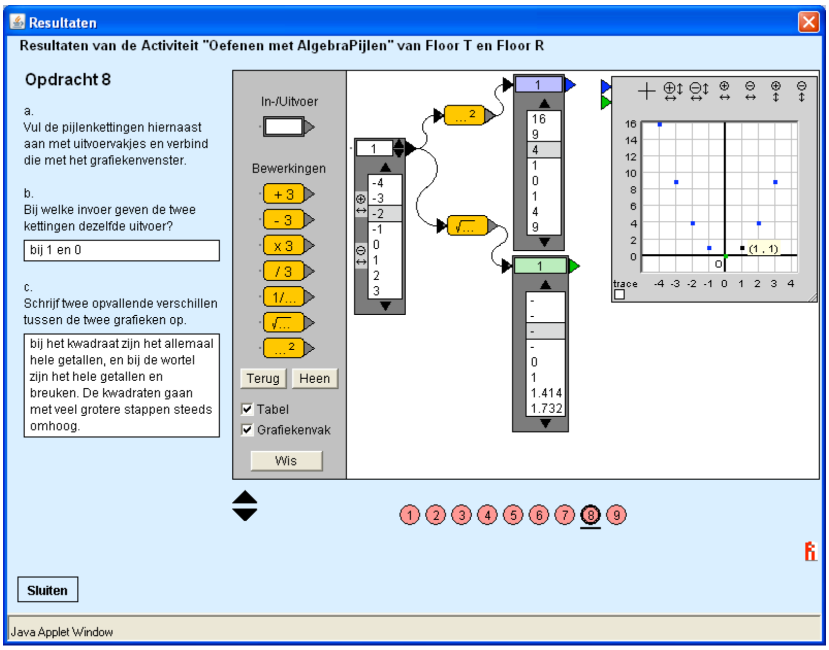

Figure 2 Revision of the answer after whole class discussion

APPLYING THEORY TO PRACTICE

In this section we apply the theory of instrumental orchestration to the above teaching episode, which essentially reflects the teacher's way to treat a misconception of (at least) one of the students, whose use of the Table-Graph technique leads to thinking it is 'special' that two points reflecting function values for the same input value are vertically aligned.

Let us call the instrumental orchestration the teacher puts into action the ‘spot and show orchestration’. By ‘spot’ we mean that the teacher, while preparing the lesson, spotted the students’ work in the ELO and thus came across Florence’s misconception. The ‘show’ refers to the teacher’s decision to display Florence’s results as a starting point for the whole-class discussion of item 8c. The teacher’s phrase “Oh yes, I actually wanted to see quite a different one” and her straight navigation to Florence’s work reveal her deliberate intention to act the way she does.

The *didactical configuration* for the preparatory phase consists of the ELO’s option for teachers to look at the students’ work at any time. As a result, the teacher notices the misconception and decides to deal with it in her lesson. This preparation is instrumented by ELO-facilities that are not available for students. In this sense, the teacher’s artefact is different from the students’ artefact. For the classroom teaching, the configuration includes a regular classroom with a PC with ELO access, connected to a data projector. Apparently, the teacher finds the computer lab not appropriate for whole-class teaching. The screen is projected on the wall above the blackboard, thus enabling the teacher to write on the blackboard, which she regularly does – though not in the episode presented here. Both the way of preparing the lessons and the setting in the classroom are observed more often in this teacher’s lessons.

The *exploitation mode* of this configuration includes putting the computer with the data projector in the centre of the classroom. This choice is driven by the constraints of one of the artefacts: if the projector was at the front, the projection would get too small for the students to read. A second choice made by the teacher is to operate the PC herself. These two aspects of the exploitation mode result in the teacher standing in the centre of the classroom, with the students closely around her, all focused on the screen on the wall. From these and other observations, we conjecture that this exploitation mode enhances classroom discussion and student involvement. Observations of another teacher using the same orchestration in a less convenient setting support this conjecture.

The *didactical performance* starts with the teacher reading the student’s answer with some minor comments (“Look here, …”). Then she reformulates the answer and asks Florence for an explanation (“What they say…”). When the explanation turns out to be inappropriate, she makes Kay give his explanation, and checks whether Florence understands it. When this is not the case, the teacher rephrases Kay’s explanation and once more checks it with Florence, who now says she understands. Of course, this didactical performance might be different a next time. For example, Florence could be asked to explain her understanding in her own words.

Now how about the link between instrumental orchestration and instrumental genesis? As the episode does not show students using the artefact, we do not see direct traces of the students’ instrumental genesis. We do claim, however, that Florence’s idea of two vertically aligned points being special is part of her scheme of using the TableGraph technique to produce point graphs. Even though this is a misconception, the episode shows that the teacher can exploit the students' experiences, and those of Florence and Kay in particular, for the purpose of attaching mathematical meaning to the technique they used, which leads to a convergence in a shared function conception in class. We see the development of mathematical meanings of techniques as an important aspect of instrumental genesis.

This 'spot&show' orchestration was one of the options suggested in the teacher guide accompanying the teaching sequence. Still, this teacher used it quite often, whereas she felt free to neglect other suggestions made in the teacher guide. In the post-experiment interview, she indicated to really appreciate the possibility to get an overview of students' results while preparing the lesson: "The ELO is practical to see what students do, you can adapt your lesson to that." She seemed to see this 'spot&show' orchestration as a means to enhance student involvement and discussion, which she believed to be relevant and seem to be part of her operational invariants . We do not have data, however, that confirm such operational invariants across other teaching settings.

Finally, an interesting aspect of the teacher's own instrumental genesis is worth discussing. The teacher points with her mouse on the screen, but does not really make changes in the students' work. Other observations suggest that she doesn't do so because she is afraid that such changes will be saved and thus affect the students' work. When she learns that this is not the case as long she uses her teacher login, she benefits from the freedom to demonstrate other options and to investigate the consequences of changes. This behaviour is instrumented by the facilities of the artefact that she initially was not aware of.

REFLECTION ON THE THEORY AND THE CASE

Let us briefly reflect on the application of the theory of instrumental orchestration to the data presented above. A first remark is that the three elements of the model – didactical configuration, exploitation mode and didactical performance – allow for a distinction and an analysis of the relevant issues within the orchestration, and their interplay. As such, the model offers a useful framework for describing the orchestration by the teacher.

As a second remark, however, we notice that it is not always easy to decide in which category something that is considered relevant should be placed. For example, does the fact that the teacher operates the computer herself belong to the didactical configuration or to the exploitation mode? This probably is a matter of granularity: if we study the 'spot & show' orchestration, this is part of the exploitation mode. If the focus of the analysis is on students' activity, it might be identified as a didactical configuration issue.

A third reflection is that the model has the advantage of fitting with the instrumental approach of students learning while using tools. This has proved to be a powerful approach (Artigue, 2002; Kieran & Drijvers, 2006), and it is therefore of great value having a framework for analysing teaching practices that is consistent with it. As such, instrumentation and orchestration form a coherent pair. In terms of instrumentation, we notice that the teachers' tasks, artefacts and techniques are not the same as those of the students; still, we can use a similar framework for analysis and interpretation.

The time dimension in the model – ranging from the didactical configuration having a strong preparatory character to the didactical performance with its strong ad hoc character – comes out clearly in the model. For the structural dimension, this is not as straightforward. As a fourth remark, therefore, we notice that operational invariants of the teacher are not limited to the preparatory phases, but also emerge in the performance. For example, the wish to have students explain their reasoning to each other appears as an operational invariant for this teacher, which is more explicit in the performance than in the configuration or in the exploitation mode. As an aside, we are aware that the data presented here do not allow for full identification of the teacher's operational invariants. More observations over time need to be included.

CONCLUSION

As far as this is possible from the one single, specific exemplary case study presented in this paper, we conclude that the model of instrumental orchestration can be a fruitful framework for analysing teachers' practices when teaching mathematics with technological tools. As it is important for teachers to develop a repertoire of instrumental orchestrations, more elaborated examples are needed. Such examples could not only help us to better understand teaching practices, but could also enhance teachers' professional development.

In addition to the need to find and elaborate exemplary orchestrations, a second challenge is to link the theory of instrumental orchestration with complementary approaches. Lagrange (2008), for example, uses additional models provided by Saxe (1991) and Ruthven and Hennessey (2002) to identify and understand teaching techniques. Another interesting perspective concerns the alternation of teacher guidance and student construction, as described by Sherin (2002). In short, the instrumental orchestration approach is promising, but needs elaboration and integration with other perspectives. For the moment, however, its descriptive power seems to be more important than its explanatory power.

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A DIDACTIC ENGINEERING FOR TEACHERS EDUCATION COURSES IN MATHEMATICS USING ICT

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A first part of our research led us to define a theoretical framework to analyse teachers' education courses and to make hypotheses to explain the lack of efficiency of teachers training (Emprin, 2007) (Emprin, 2008). This paper presents the continuation of this work. We use the methodology of didactic engineering, adapted to teachers' education, and a theoretical framework previously built to test our hypothesis. In a first part of this paper we describe our theoretical framework and hypothesis about teachers training. In a second part we develop the didactic engineering and its results.

TEACHERS EDUCATION COURSES ANALYSIS

The general question guiding this work is the difficulty for mathematics teachers to use ICT in their classrooms. Our choice is to focus our attention on a particular factor explaining this difficulty: teachers' professional education; without denying the existence of other factors such as material problems, resources available etc. Several studies in France as specified in DPD (2004), wider as Empirica (2006) or TIMSS & PIRL (1995) and reports as Jones (2004) indicate this explanatory factor. French political choices since 1970[5] show that a quantitative effort was made, our research thus relates to a qualitative problem of teachers' training.

A theoretical framework

First we chose to use a framework designed for the analysis of teaching practices and to specify it with teacher educators' practices: the two-fold approach. This framework, defined by Robert (1999), Robert & Rogalski (2003) does not take into account specifics of the use of technology. This leads us to use, jointly with the two-fold approach, a framework making it possible to take into account this dimension as described in (Emprin, 2008). The instrumental approach developed by Rabardel (1995) (1999) appears to be relevant. This approach, which was already developed in the didactic of Mathematics (Artigue, 2002b), (Trouche, 2005), leads us to analyse instrumental genesis.

One difficulty is that teacher educators' practices can not be reduced to a teaching activity. A teachers' educator, in France, was most of the time a secondary school teacher, in many instances they keep on teaching to pupils. For this reason, like

[5] In this year began the IPT plan (Informatique Pour Tous) which could be translated in "data computing for everyone". For example in 1985, it allowed the purchase of computers for 33.000 schools and represented 5.500.000 hours of training for teachers (Archambault, 2005).

Abboud Blanchard (1994) specifies, the teacher trainer's previous practices as a teacher intervene in his practices as a teachers' educator.

We borrow the definitions of "activity" and "practices" from Robert & Rogalski (2002) which we must specify on various levels met during a teachers' education course:

This definition of "activity" is nearly similar to Rabardel's notion of "productive activity" (Rabardel, 2005, p. 20). It contains actions but also statements, attitudes and unobservable aspects which influence actions.

The definition of "practices" we use is a reconstitution of the five components described in the two-fold approach. Robert & al. (2007) give the description we have translated here:

> "We developed, taking into account the complexity of the practices, analyses capable of giving an account of what can be observed in class, which results from teacher's homework and the unfolding, and factors which are external to the classroom but which weigh on practices, including those in the classroom, and eventually contribute to the teachers' choices before *and* during the lesson. Indeed, practices in classroom are forced, beyond goals in terms of pupils' acquisitions, by determinants related to teachers' trade: institutional, social… Let us quote programs, timetables, schools, colleagues, class and its composition. Moreover, the practices have a personal anchoring which refers to the teacher as a singular individual, in terms of knowledge, picturing, experiments, trade's idea and also conditions its choices. Our analyses start from class session in which we distinguish components, institutional, social, personal, meditative (related to the unfolding in the classroom and improvisations), cognitive (related to the prepared contents and expected unfolding), closely dependent for a given teacher, and having to be recomposed: it is necessary for us to think of the components together, and to estimate the compensation, balance, the compromises to include/understand and start to explain what is concerned. »

To build our framework of analysis we need to dissociate the various levels of activities and practice but also to see their interactions. Figure 1 makes it possible to describe these various levels.

The first level of activity is that of the pupil. We note it A0 level. The pupil has a task to realize, and acts accordingly. He uses an instrument belonging to ICT. This level can thus be analyzed with the didactic of mathematics and the instrumental approach. The observation of the process of instrumentation/ instrumentalisation informs us about the instrumental geneses of the pupil and the instruments built.

The second level of activity is that of the teacher whom we note at level A1. The tasks of the teacher consist of managing and organizing the activity of the pupils. He also organizes the instrumental geneses of the pupil. The two-fold approach enables us to analyze a first level of practices which we note P1 level.

The other two levels of activity are those which exist in teachers' education courses. The activities of the trainees (who are thus teachers) during the training course, are noted as A2. They are organized by those of the teachers' educator noted as A3. The Two-fold approach and the instrumental approach give us access to a second level of practices noted as P2, those of the teachers' educators.

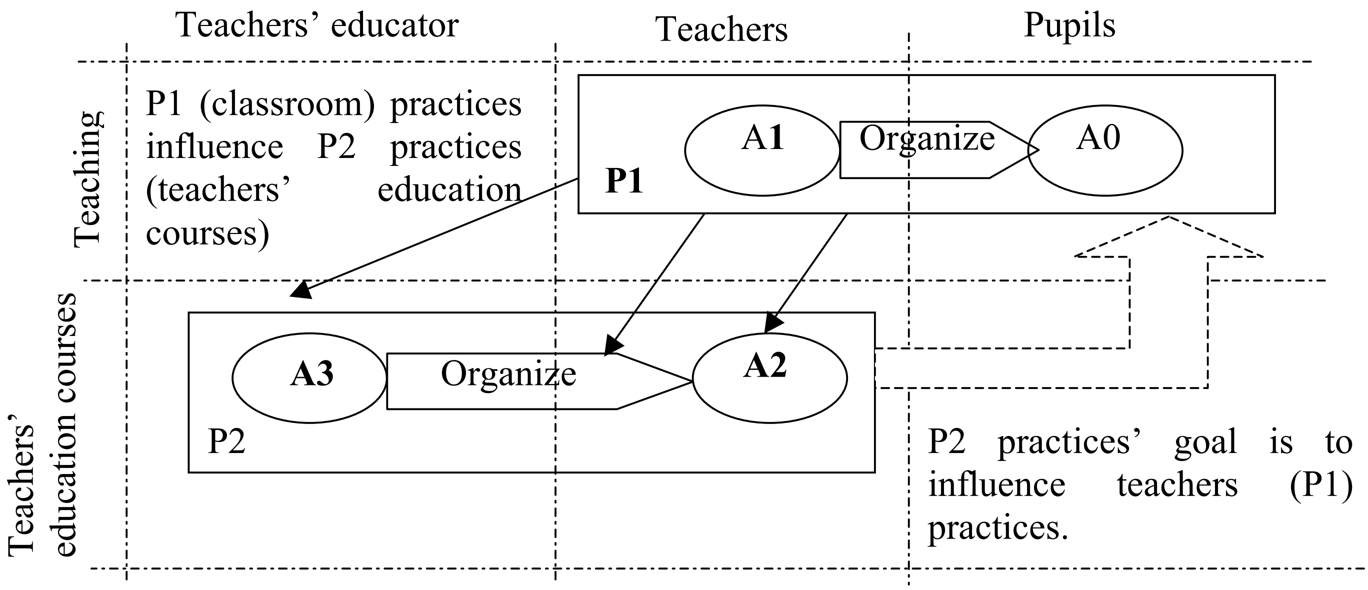

Figure 1: overlap of the four levels of activity and two levels of practices

Use of the theoretical framework

Our work is centred on the analysis of the teachers educators practice, thus we neither directly analyze the practices of P1 level, nor activities of A1 and A2 levels, nevertheless they appear during teachers' education courses as explanatory factors.

Teachers practices (P1) can be seen during teachers education courses in three main ways, through a video: practices are shown, when the teacher's educator narrates a classroom session: practices are narrated through what the teacher's educator asks the trainees to do: the practices are inherent. This last way is linked with a strategy of teachers training which is called homology. This strategy described by Houdement & Kuzniak (1996) shortly consists in doing with teachers (A3→A2) what they will be expected to do when they are back in their classrooms (A1→A0).

Two-fold approach is designed to analyze the real practices; it requires being able to observe the courses and to ask the teacher about the context in which he works. To analyse P1 practises which appear during teacher education session we use two-fold and instrumental approaches as a reading grid to see which part of practices teachers' educator focuses on.

Hypothesis resulting from the analysis of teachers education courses

We implemented this framework of analysis in Emprin (2007) on a corpus of three teachers' education courses, of fourteen interviews of teachers' educators. The results obtained help us to build the first part of our hypothesis about the lack of effectiveness of teachers trainings.

First we notice that the working time is mainly dedicated to a work on computers (more than 50% of the time). When trainees are not in front of computers, the time is devoted to explanations (44 to 62%) and descriptions (35 to 52%) given by the teachers' educator, there is thus very few analysis or debate. In term of two-fold approach social, personal and institutional components of the practices are almost not approached. The mediative component of practices appears in the analysis of video or the narration of courses, but is not questioned. The cognitive dimension remains rather marginal. Our analysis also shows a possible drift of homology strategy: it is likely to introduce confusion between the various instrumental geneses, of pupil and teacher.

BUILDING OF A DIDACTIC ENGINEERING FOR TEACHERS EDUCATION COURSES

Hypothesis

We identify two complementary ideas explaining the lack of efficiency pointed previously. The first one results from the work of Ruthven & Hennessy (2002) and Lagrange & Dedeoglu (in preparation). Theses authors show a gap between teachers' needs and ICT potentialities presented by teachers' trainers. We also observe an absence. In France the "reflexive practitioner" of Schön (1994) and the "analysis of practices" developed by Altet (1994) (2000) or Perrenoud (2003) are two important models for teachers' education is thus remarkable that no allusion is made there in teachers' education courses to mathematics with ICT. That leads us to consider the introduction of a reflexive component in ordinary practices' analysis and to formulate four hypotheses taking into account the first part of our work:

- The analysis of real practices would make it possible to initiate a reflexive attitude in teachers (making it possible for the teacher to change their teaching practices)
- Leading trainees to analyze a real professional problem enables them to confront their representations, mobilize their knowledge (resulting from experience) and come to a consensus based on reasoning.
- An analysis of the professional practices taking into account several dimensions of practices (in terms of two-fold approach) and based on the analysis of the relationship between teaching practices and activity of the pupil, makes it possible for the trainees to mobilize their knowledge (resulting from experience and their theoretical knowledge).
- It is necessary to contribute, during teachers' training courses, to the professional instrumental geneses of teachers and to analyze the lessons in terms of instrumental needs and potential instrumental genesis of pupils.

In order to check these hypotheses we use the methodology of didactic engineering that we specify to teachers' education. This methodology defined by Chevallard (1982) and Artigue (2002a) is based on the verifying of a priori hypothesis. Thus we need to define observable criteria linked to our hypothesis. We decline our four hypotheses in seven criteria:

- The trainees' ability to identify and define a problem.
- The formulation and the use, by the trainee, of knowledge coming from experience associated with theoretical knowledge to analyze the practice
- The implication of trainees' personal practices and of his own experience in the analysis.
- The trainees reach a consensus based on knowledge coming from experience and theory.
- During the session teachers' educator do not give any answers, any explanations. The knowledge is built by trainees and not given by teachers' educator. We call that an a-didactical lesson referring to theory of didactical situations (Brousseau, 1998)
- The fact that the analysis makes it possible to take into account several dimensions of the practices
- It must then be possible to identify trace of instrumental genesis making it possible for teachers to consider instrumented actions but also results on pupils' activity.

Our methodology leads us to conceive a scenario for teachers' education whose implementation will be analyzed by means of the theoretical framework built in the first part.

Scenario and analyzes

The scenario is inspired from Pouyanne & Robert (2004) (2005). It is based on the analysis of teaching practices by means of a video. Four periods are defined: an a priori analysis of the lesson (which has been recorded) where hypothesis about the effects of the teaching practices on pupils' activity are put forward; an analysis of the video and a comparison with the hypothesis; a search for alternatives based on the question "What would you do if you had to do such a lesson?"; and finally a debate around problems emerging during the first three period.

We implemented this scenario twice, in each one, videos show pupils using interactive geometry software (IGS): In the first training course eight grade pupils had to prove that perpendicular bisectors in a triangle converge. The second video show sixth grade pupils solving a problem (which is detailed below). We develop now this second session of teacher education.

In each teacher's education session, the scenario takes about three hours. This part of the session have been recorded, transcribed and analysed. The analysis takes into account who is speaking, the type of speech (description, explanations, analysis) and it's content.

An example of session

The lesson recorded for this teachers training is what we call in French "an open problem" referring to Arsac & Mante (2007). This type of problem is called "open" insofar as no specific solution is expected: what matters is pupils' search.

Figure 2 gives the statement of the problem. Pupils are asked to say which one of [EG] or [AC] is longer.

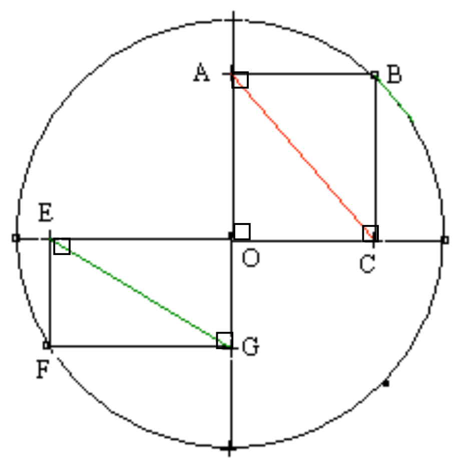

Figure 2 : an « open problem »

During the first part of the work with trainees, the a priori analysis, we had to let them use the IGS. It is a first change in the scenario. It seems to be very difficult for teachers to analyse the problem without having a working time on the computer. This time is not a time of homology even if the trainees do what is expected from pupils.

During the analysis the trainees have a transcription of the discussion with the teacher who is in the video. She specifies what is at stake in this lesson: she wants pupils to develop their critical thinking and to show them not to trust their perception. The trainees identify three stakes: the drawing with the software, the location of the rectangles in the whole geometrical drawing and the property of the diagonals of a rectangle. They specify that they think that the situation is not feasible with pupils. They propose teaching aids to make the situation feasible. They propose to reveal the radius of the circle, the other two diagonals of the rectangle. Another solution considered is to cut out the problem or to make a preliminary recall of the useful properties. In this stage there is thus an implication of the trainees who adapt the lesson since they try, to some extent, to make it feasible in their classrooms. This implication can be seen in the following example.

> Trainee: that seems difficult to me in 6th grade also because I think that they will see that the diagonals have the same length but that they will not be able to justify it.

The viewing of the film reveals initially the need for dissociating the task of construction in the software from the remainder. Indeed the pupils encounter real difficulties to build the geometrical figure. The trainees realize that pupils need to build uses of the software. It is a part of the instrumental genesis. On the video, once geometrical construction has been carried out, the pupils try to conjecture. The trainees realize that pupils have the necessary knowledge to solve the problem but that they are not able to mobilize it.

In the film, the pooling of pupils' works take place at the end of the lesson, whereas the pupils are still in front of the computers. It is quickly carried out by the teacher. The conclusions of the trainees are that it is necessary to take more time, to move the pupils away from the computers and to let them talk. There is thus a clear evolution in the trainees' mind. In the first part of the analysis they have doubts about the ability of the pupils to solve the problem and in the last part they say it is necessary to devote more time to the pooling of what pupils have found.

The search for alternatives contains the essential components of the analysis. The trainees reaffirm that it is necessary to dissociate the drawing on IGS from conjecture. Some even propose to remove the drawings' work. This work also allows a long discussion about the place of this problem in pupils' training. Before pupils know the property of the diagonals of the rectangle, the problems is centred on research whereas afterwards it acts more as a consolidation of knowledge. That also leads to discuss the place of observations in the geometrical trainings. A trainee proposes to use this problem to introduce the property of equality of the diagonals which disturbs another trainee who believes that observing properties is conflicting with the idea of mathematics. This trainee finally realizes that she does not have tools to give proof of the property to pupils of this level while at the same time the property is in the official programme. During these discussions the teachers' educator almost does not intervene. Trainees are personally involved in the analysis:

> Trainee: I do think that giving the instructions when the computers are "on" is always rather difficult; it is better to give instructions before turning the computers on.

In this example we can see that this trainee formulates a teaching knowledge, rather simple but which can now be used consciously by other trainees.

Most of the indicators can be observed for "many" trainees. Nevertheless, during a three hours session, a limited number of trainees can speak and consequently the internal evaluation of our methodology is only partial.

Finally, we noticed two changes in our scenario: the time of appropriation of the software was introduced during the analysis of the lesson and the final time of debates was removed. For the first change, the lack of acquaintance of the trainees with the artefact prevents them from making a real analysis. The second change is due to time devoted to debates during the session. The entire subject likely to be alluded to seems to have been discussed before. A last noticeable point is that trainees do not know other pieces of software which could be used in this lesson. The teachers' educator had to show different pieces of software as in the teachers' education courses we analysed in the first part of our work.

Conclusion on the didactic engineering of formation and continuation

The main results of this didactic engineering are linked with our criteria: it seems to be necessary to let the trainees use and try the artefact. It helps them to analyse the lesson but it also seems to match with trainee expectations. It is possible to take into account several dimensions of the practices but in a smaller number than expected. The analysis of the video helps trainees to make cognitive and mediative components more explicit but the other components are more difficult to reach. The scenario built allows a reflexive analysis of the practices. Experience and theoretical knowledge is used to analyze the problem of introduction of the ICT. Instrumental geneses of the teachers and the pupils are really dissociated. The trainees considered what is necessary to pupil to use ICT in this lesson. They also found different options and they analysed the changes involved by these choices in term of learning or in lesson unfolding. For example asking pupils to draw the figure in the software helps them to use a proper vocabulary (because the software makes it compulsory) but it takes a long time and leads the teacher to reduce the time of conjecture.

Practices, in our didactic engineering, are shown in a video but it is possible to work on other types of practices such as real practices or simulated practices as the work of Morges (2006) suggested. Simulated practices make it possible for a whole group of trainees to work on the same teaching experience. The construction of such a simulator is the object of a work we initiated in 2007.

To conclude, the fact that teachers use experience knowledge to analyze practices with ICT makes it possible for us to consider the teachers' education course with ICT as a lever for teachers' education generally speaking. It seems to be easier to influence the way of teaching mathematics by influencing the way of teaching mathematics with ICT.

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LEADING TEACHERS TO PERCEIVE AND USE TECHNOLOGIES AS RESOURCES FOR THE CONSTRUCTION OF MATHEMATICAL MEANINGS

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This paper presents the early results of an on-going research project on the use of technology in the mathematics teaching and learning processes. A first aim of this project is to understand how deeply math teachers do perceive the opportunities technologies can bring about for change in pedagogical practice, in order to effectively use them for the students' construction of mathematical meanings. Secondly, the research aims at verify if teachers realise that, in order to successfully deal with perturbation introduced by technologies, they have to keep themselves continuously up-to-date and to acquire not only a specific knowledge about powerful tools, but also a new didactical and professional knowledge emerging from the deep changes in teaching, learning and epistemological phenomena.

INTRODUCTION

Due to the continuous spread of technology in the latest years, challenges and expectations in the everyday life, and in education in particular, have dramatically changed. Within this context of rapid technological change the world wide education system is challenged with providing increased educational opportunities. The use of Information and Communication Technology (ICT) in the classroom, however, seems to be, in the majority of cases, still based on traditional transfer model characterised by a teacher-centred approach (see for example: Midoro, 2005).

But, according to Holyes et al. (2006; p.301):

> "…a learning situation had an economy, that is a specific organization of the many different components intervening in the classroom, and technology brings changes and specificities in this economy. For instance, technological tools have a deep impact on the "didactical contract…".

That is, the technology-rich classroom is a complex reality that necessitates observation and intervention from a wide range of perspectives and bringing technology in teaching and learning "adds complexity to an already complex process" (Lagrange et al. 2003).

Moreover, as underlined by Mously et al. (2003; p.427),

> "technological advances bring about opportunities for change in pedagogical practice, but do not by themselves change essential aspects of teaching and learning".

As research underlines (Bottino, 2000), indeed, innovative learning environments can result from the integration among educational and cognitive theories, technological opportunities, and teaching and learning needs. However, it is extremely important for teachers to confront themselves with the necessity to understand how the potential offered by technology can help in the overcoming of the everyday didactical practice complex problems.

I believe that for technologies to be effectively used in classroom activities teachers need, not only to "accept" the presence of technologies in their teaching practice but also to see technologies as learning resources and not as ends in themselves. Moreover, learning activities involving technologies should be properly designed to build on and further develop mathematical concepts. Hence, an "adequate" preparation is essential for teachers to cope with technology-rich classrooms, so that using computers not merely consists on a matter of becoming familiar with a software.

This paper presents the early results of an on-going research project on the use of technology in the mathematics teaching and learning processes, investigating mathematics teachers' perceptions of ICT and of their usefulness in promoting a meaningful learning.

A first aim of this project is to understand how deeply math teachers, both pre-service and in-service, do perceive the opportunities technologies can bring about for change in pedagogical practice in order to effectively use them for the students' construction of mathematical meanings.

Secondly, the research aims at verify, whether or not, teachers realise that, in order to successfully deal with perturbation introduced by technologies, they have to keep themselves continuously up-to-date and to acquire not only a specific knowledge about powerful tools, but also a new resulting didactical and professional knowledge emerging from the deep changes in teaching, learning and epistemological phenomena.

THEORETICAL FRAMEWORK AND RELATED LITERATURE

Many researchers in the latest years are answering the challenge to provide educational opportunities by studying teaching and learning mathematics with tools (Lagrange et al., 2003).

Results of both empirical and theoretical studies have also led to the elaboration of the idea of "mathematics laboratory" as reported, for example, in an official Italian document prepared by the UMI (Union of Italian Mathematicians) committee for mathematics education (CIIM):

> "A mathematics laboratory is not intended as opposed to a classroom, but rather as a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects" (UMI-CIIM MIUR, 2004; p.32).

In this sense, a laboratory environment can be seen as a Renaissance workshop, in which the apprentices learned practicing and communicating with each other. In particular in the laboratory activities, the construction of meanings is strictly bound, on one hand, to the use of tools, and on the other, to the interactions between people working together (without distinguishing between teacher and students).

According to this approach, and as in Fasano and Casella (2001), I believe that technological tools can assume a crucial role in supporting teaching and learning processes if they allow teachers to create suitable learning environments with the aim to promote the construction of meanings of mathematical objects.

Moreover, in agreement with this point of view, I consider important to highlight that, again quoting the UMI-CIIM document (p.32):

> "The meaning cannot be only in the tool per se, nor can it be uniquely in the interaction of student and tool. It lies in the aims for which a tool is used, in the schemes of use of the tool itself. The construction of meaning, moreover, requires also to think individually of mathematical objects and activities."

Furthermore, as claimed by Laborde (2002; p.285),

> "whereas the expression integration of technology is used extensively in recommendations, curricula and reports of experimental teaching, the characterisation of this integration is left unelaborated".

In particular, she underlines the idea that the introduction of technology in the complex teaching system produces a perturbation and, hence, for teacher to ensure a new equilibrium he/she needs to make adequate, non trivial choices. Integrating technology into teaching takes time for teachers because it takes time for them, first of all to understand that, and how, learning might occur in a technology-rich situations and, then, to become able to create appropriate learning situations. This point of view is based on the idea that a computational learning environment could promote the learners' construction of situated abstractions (Noss & Hoyles, 1996; Hölzl, 2001) and on the "instrumental approach" as developed by Vérillon and Rabardel (1995).

Within the instrumental approach, the expression "instrumental genesis" has been coined to indicate the time-consuming process during which a learner elaborates an instrument from an artefact: it is a complex process, at the same time individual and social, linked to the constraints and potential of the artefact and the characteristic of the learner. If, according to the instrumental approach, learners need to acquire non-obvious knowledge and awareness to benefit of a instrument's potential, I firmly believe that teachers need to take charge of student's instrumental genesis (Trouche, 2000).

Finally, I consider worthy of note the concept of "instrumental orchestration" proposed by Trouche (2003) aiming at tackling the didactic management of the instruments systems in order to conceive the integration of artifacts inside teaching institutions. In particular, he underlines that pre-service and in-service teacher training should take in account the complexity of this integration at three levels:

- a mathematical one (new environments require a new set of mathematical problems);

- a technological one (to understand the constraints and the potential of artifacts);

- a psychological one (to understand and manage the instrumentation process and their variability). (p.798)

METHODS, CONTEXT AND PROCEDURE

The research I'm going to present consists in two main phases. The first has been carried out with a rather small group of in-service teachers at the University of Bari and a larger group of pre-service teacher at the University of Basilicata. The second involved another small group of pre-service teachers at the University of Bari.

Teachers belonging to the first group at the University of Bari were 16 high-school teachers. Although some of them already taught mathematics, on the whole they were qualified to teach related subject and they were attending a training program in order to get a formal qualification to teach mathematics.

At first, a preliminary anonymous questionnaire was submitted to them with the aim to know if they were able to see technologies as learning resources, as well as if they were available to continuously bring up-to-date in order to properly design and manage with technology-rich classroom activities. Key questions in the questionnaire included the following:

1. Do you think ICT could be useful for your teaching activities? Why?
2. Do you think that the use of ICT can somehow change the learning environment? And the way to teach? And the dynamics among actors in the teaching/learning situations?
3. Which difficulties do you think can be encountered when designing and developing a math lessons using somehow ICT?
4. As a teacher, do you think you need to have some didactical competences in order to properly use ICT? Eventually, which ones? And anyway, why?

Within the training program they attended, a thirty hours course was focused on didactical reflection aiming at helping student teachers to understand how to make the most of the use, in mathematics teaching and learning activities, of general tools such as spreadsheets, multimedia and Internet, as well as mathematics-specific educational software such as Cabri. In order to explain them that the changes produced by the introduction of a technological tool will not necessarily per se bring the students more directly to mathematical thinking, particular attention was devoted to stress the role of the a-didactical milieu in authentic learning situations, as in the known Brousseau's (1997) "theory of didactical situations". Furthermore, they were asked to analyse and discuss both successful and questionable examples of teaching/learning mathematics activities in which an important role has been played by the use of ICT.

At the end of the course student teachers designed a teaching/learning activity involving somehow the use of technology: in this way I intended to verify how deeply they have perceived the opportunity to effectively exploit the usage.

A further anonymous questionnaire, free from constraints, was later submitted with the aim to find out any signal for changes in their conceptions to have been occurred. Key questions in this further questionnaire were exactly the same.

Pre-service teachers involved in the research project at the University of Basilicata were a larger number (97). They were only asked to fill in the first questionnaire.

During the second phase, a group of 16 pre-service teachers at the University of Bari, instead, interacted with the researchers/educators in the same way of the first group of in-service teachers: to this further group of teachers a preliminary anonymous questionnaire was submitted; then, they were invited (during a thirty hours course) to reflect on didactical aspects of the use of technologies as well; at the end of the course they were asked to design a teaching/learning activity in which technology played an essential role; finally I analysed the extent of their changes in looking at the integration of technologies in the teaching/learning processes.

According to the results obtained during the first phase (that I'm going to present and discuss in the next paragraph), in the second phase I asked student teachers, not only to design a teaching/learning activity involving the use of technology, but also to put in action the activities they have designed, having as student sample their colleagues: in this way they proved themselves as "actors" in a technology-rich learning "milieu".

FINDINGS AND DISCUSSION

Findings from the first anonymous questionnaire revealed that in-service student teachers perceived that technology can bring support to their teaching (see Fig.1), but only as much as it is a motivating tool enabling students understanding per se (see Fig. 2).

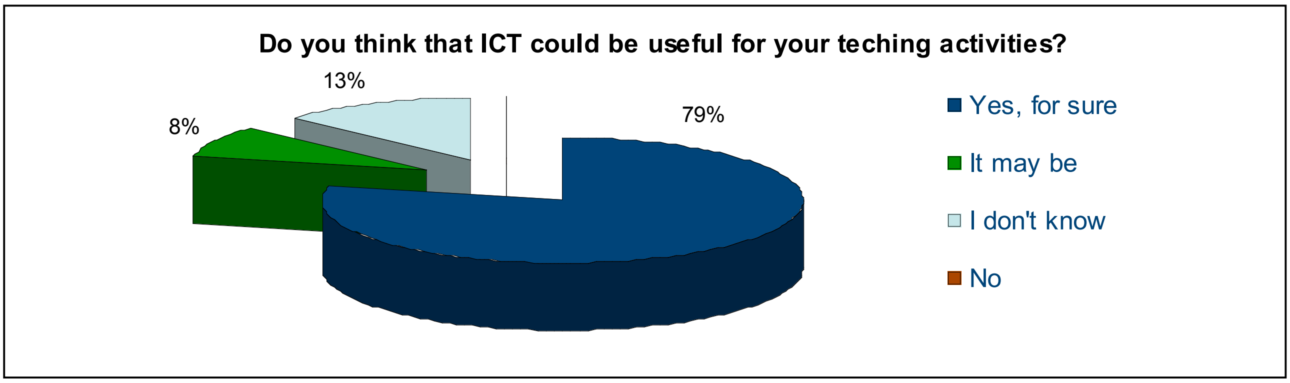
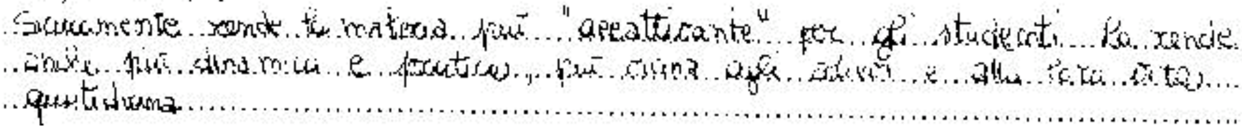
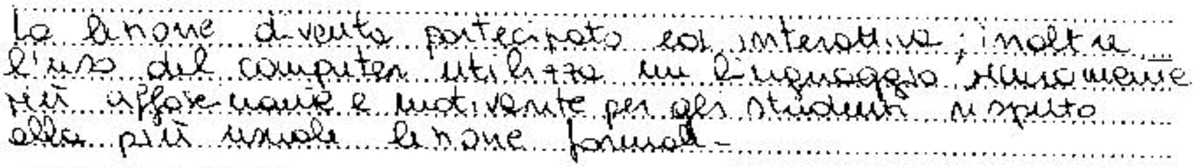


Figure1: The 79% of the in-service student teachers gave a positive ("Yes, for sure") answer to question 1.

Figure2: Some in-service student teachers' answers to question 1: Do you think ICT could be useful for your teaching activities? Why?

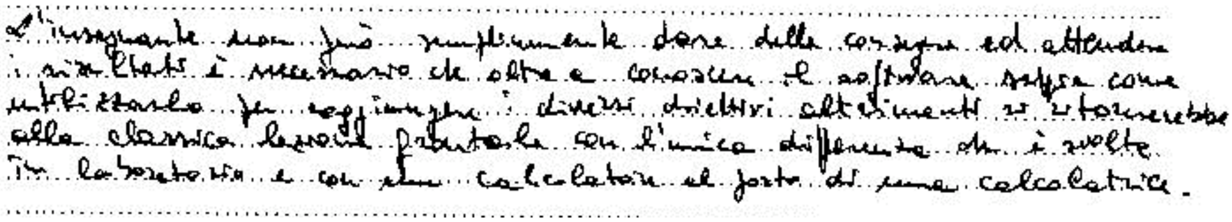

Sicuramente rende la materia più "accattivante" per gli studenti. La rende anche più dinamica e pratica, più vicina agli alunni e alla loro vita quotidiana.

…math can be more attractive, dynamic, practical


La lezione diventa partecipata ed interattiva; inoltre l'uso del computer utilizza un linguaggio sicuramente più affascinante e motivante per gli studenti rispetto alla più usuale lezione frontale.

…lesson can be more, shared, interactive, fascinating

Answers given by the pre-service teachers were, instead, a little bit more didactically oriented: some of them recognise that, if nothing else, the knowledge of the instrument functionality is probably not enough for a teacher to use it in an effective way in terms of construction of meanings by the students (see Fig. 3).


L'insegnante non può semplicemente dare delle consegne ed attendere i risultati. È necessario che oltre a conoscere il software sappia come utilizzarlo per raggiungere i diversi obiettivi altrimenti si ritornerebbe alla classica lezione frontale con l'unica differenza che è svolta in laboratorio e con un calcolatore al posto di una calcolatrice.

…otherwise the only difference with the classical lesson would be the use of a PC instead of a calculator

Figure3: A pre-service student teacher's answer to question 1.

None of the in-service teachers recognised that technology could bring a great support in creating new interesting and attractive learning environments. While, at least some interesting observation could be revealed among answers given (to question 2) by the pre-service teachers: some of them suggested the use of technological tools to allow students "collaboratively solve intriguing problems".

Be aware of the opportunity to create a new "milieu" and change the "economy" of the solving process was, however, extremely far from their perception of the use of technology in mathematics teaching/learning activities, both for in-service and for pre-service teachers.

About the question 3, concerning the difficulties they think can be encountered when designing and developing a math lessons using somehow ICT, they mostly ascribed possible difficulties to the lack of an adequate number of PC and the technical problems that might occur, but also to the natural students' bent for distraction and relaxation, especially when facing a PC (see Fig. 4).

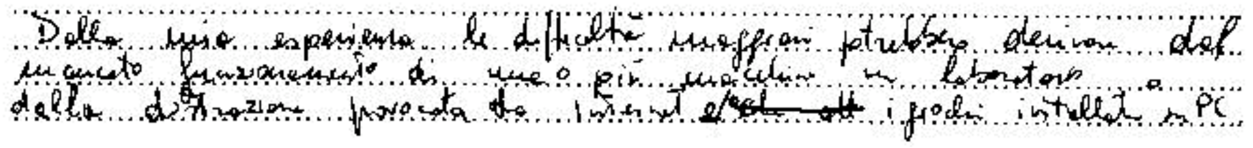


Figure4: Some student teachers consider new technology as a motivating tool that requires motivation.

As a consequence they did not feel the need to be skilled in using technology for their teaching and did not usually consider that their lack of skills presents them with any difficulties. And, although the 75% of the student teachers recognised (answering to question 4) the need to have some didactical competences in order to use new technology, what they asked to know about was, in most of the cases, just software functionalities (not potential, nor constrains): only some of the pre-service teachers also asked to know how to effectively integrate their use in the teaching practice.

Even tough some of the activities that in-service teachers prepared at the end of the course revealed the willingness to attempt a new approach to the use of ICT, answers to the second anonymous questionnaire shown they still continued to find difficulty to be aware of the potential offered by ICT (see Fig. 5).

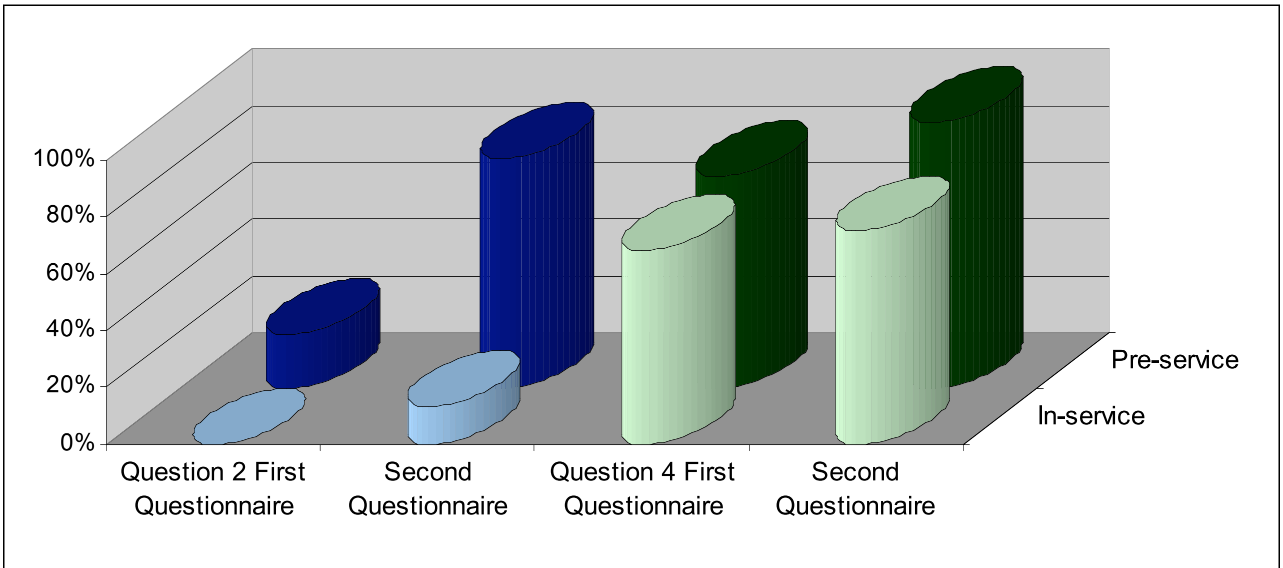


Figure5: Percentage of positive ("Yes, for sure") answers given by both in-service and pre-service teachers respectively to the first and the second questionnaire to questions 2 and 4.

For this reasons, for the second phase of the project I planned to pay particular attention to promote teachers' reflections on the opportunities offered by appropriate uses of technological tools in order to create new learning environment and, according to the idea of "mathematics laboratory", to foster the construction of mathematical meanings.

Student teachers were invited not only to design a possible teaching/learning activity involving somehow the use of technology, but they were also involved in a “mise en situation” (as in the known Chevallard’s approach) during which they had the opportunity to assume the roles of the student, the teacher and a researcher/observer.

In this way, they faced with the complexity of the integration of technologies in classroom practice. Their comments at the end of the experience shown that they have developed an awareness of how the students’ instrumental genesis can take shape (psychological level). Moreover, answers to the second anonymous questionnaire revealed that they felt the need to understand the constraint and the potential of technologies (technological level) and to look for new mathematical problems (mathematical level).

EARLY CONCLUSIONS AND FUTURE WORKS

Discussion suggested by the researches in this field and by the analysis of this on-going experience led me to reflect on and to underline that an adequate preparation is essential for teachers to cope with technology-rich classrooms. In particular I believe that, only if teachers become aware of the potential usefulness and effectiveness of technologies as methodological resources (enable to foster the constructions of meaningful learning environment) they would recognise the need of an effective integration of them in the classroom activities and view new technologies as cultural tools that radically transform teaching and learning.

At the actual stage of this on-going research I can claim that, in my opinion, most of the teachers have difficulty to acquire the awareness of the potential of technology as a methodological resource. Hence, as educators, we also have to deal with the need to lead teachers to develop a more suitable and effective awareness of the usage of new technologies. Furthermore, I believe that the difficulty teachers have to acquire this awareness could be overcome giving teachers the opportunity to be subject of a “mise en situation”. In this way teachers can experience by themselves the difficulties students can encounter and have to overcome, the cognitive processes they can put in action and the attainment they can achieve. They also have the opportunity to understand and manage with the students’ instrumental genesis and to become more skilful and self-confident when deciding to exploit the potentials of technologies in mathematics education.

For future works I think in particular to go on with this idea, promoting further experiences of “mise en situation” according to the following stages:

- let teachers experience the importance of the relationship between the specific knowledge to be acquired by the students and the knowledge teacher possesses of it;
- let teachers experience the importance of the relationship between the specific knowledge to be acquired by the students and whatever students already know;
- let teachers experience the importance of the relationship between their knowledge and the students’ ones.

I suppose, indeed, that through these stages, teachers could experience by themselves the processes that come into play bringing technology in a teaching/learning situations. In particular, according to the early results of this study, I believe that in this way teachers do tackle with the obstacles encountered, the difficulties to be overcome, the cognitive and metacognitive processes carried out and the attainment that can be achieved.

To conclude, in the next future I aim to verify that, thanks to this methodology, not only they can cope with changes they could meet in a technology-rich learning situation but, reflecting on them, they can also become aware of how to better make use of technology as a resource to create an effective and meaningful learning environment.

Finally (also considering the explicit suggestions of the WG7 call for papers), I suppose that an interesting help to foster the development of teacher's instrumental genesis can be given by the use of Geoboards (Bradford, 1987). A Geoboard is a physical board (often used to explore basic concepts in plane geometry) with a certain number of nails half driven in, in a symmetrical square, (for example five-by-five array): stretching rubber bands around pegs, provide a context for a variety of mathematical investigation about concepts and objects such as area, perimeter, fractions, geometric properties of shapes and coordinate graphing.

Thus, I would like to let high school teachers operate with an unusual (at that level) context/tool like a Geoboard, and try to understand if, in this way, they can perceive teaching resources, both digital or not, as methodological resources: when teachers become aware that some resources can be effectively used for the construction of mathematical meanings they can start to successfully design and experiment new interesting learning activities.

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THE ROBOT RACE:

UNDERSTANDING PROPORTIONALITY AS A FUNCTION WITH ROBOTS IN MATHEMATICS CLASS

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This paper presents and discusses the use of robots to help 8th grade students learn mathematics. An interpretative methodology was used and data analyses were supported by Situated Learning Theories and Activity Theory. These tools allowed the accurate description and analysis of student's practices in mathematics classes. The results indicate that the use of robots to study proportionality as a function aided and supported student learning.

INTRODUCTION

Educational systems the world over are investigating new and engaging mechanisms in order to better present complex concepts and challenging domains such as mathematics. The implementation and exploration of technologies in classrooms is a promising general approach. However, we should not neglect the real world where the actual students live – a world more and more dependent on technologies. Consequently, it is essential to combine computation aids and new educational aims with a redefinition of teaching processes and teachers role's in the classroom. It is in this context that the project DROIDE was initiated in 2005.

DROIDE[7]: "Robots as mediators of Mathematics and Informatics learning" is a project with three main objectives:

(1) to create problems in Mathematics Education/Informatics areas which are suitable to be solved using robots; **(2)** to implement problem solving using robotics at three points in the educational system: mathematics classes at K-9 and K-12 levels; Informatics in K-12 levels; Artificial Intelligence, Didactics of Mathematics and Didactics of Computer Science/Informatics at the university level; **(3)** to analyze and understand students' activity during problem solving using robots.

This paper discusses research on the second issue (the implementation of problem solving using robots in mathematics class) at the K-9 level. It addresses the following research problem: to describe, analyse and understand how students learn

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mathematics using robots as mediators of learning. It particularly focuses on the mathematical concept of proportionality as a function.

THEORETICAL BACKGROUND

The research approach is derived from Situated Learning Theories (Lave & Wenger 1991, Wenger, 1998, Wenger et al, 2002) and Activity Theory (especially the 3rd generation introduced by Engeström, 2001). A key element of Situated Learning theories is the notion of a community of practice and the suggestion that learning is a situated phenomenon. In this paper, this viewpoint is used to reflect upon emergent learning within students' mathematical practices.

The Concept of Practice

According to Wenger, McDermott and Snyder (2002) practice[8] is constituted of a set of "work plans, ideas, information, styles, stories and documents that are shared by community members" (p.29). Practice is the specific knowledge that the community develops, shares and maintains. Practice evolves as a collective product integrated in participants' work and the organisation of knowledge in ways that are useful and reflect the community's perspectives (Matos, 2005).

Wenger (2002) proposes three dimensions in which practice is the source of coherence in a community: mutual engagement, joint enterprise and shared repertoire. Mutual engagement is a sense of "doing things together"; the sharing of ideas and artefacts, with a common commitment to interaction between community members. Joint enterprise is having (and being mutually accountable for) a communal common goal, a procedure which rapidly becomes an integral part of practice (Matos, Mor, Noss and Santos, 2005). Shared repertoire refers to a set of agreed resources for discussions and negotiations. This includes artefacts, styles, tools, stories, actions, discourses, events and concepts.

The Concept of Mediation

Engeström (1999) conceptualizes an activity model formed by three elements – the subject, the object and the community – with mediation relations between them. In the context of this research, the mathematics classroom forms such an activity system. The subject is part of a collective; reflecting the fact that we do not act individually in the world. The subject is part of a system of social relations.

The concept of mediation has a central role in Activity Theory[9]. It is based on the presupposition that the subject does not act directly on the environment; that it has no

[8] The term practice is sometimes used as an antonym for theory, ideas, ideals, or talk. In Situated Learning theories that is not the idea. In Wenger's sense of practice, the term does not reflect a dichotomy between the practical and the theoretical, ideals and reality, or talking and doing. The paper extension does not allow the development of the idea of practice. For discussion of practice related with mathematics education see Fernandes (2004).

[9] For a more general vision of Activity Theory see http://pparticipar-t-act.wikispaces.com/

direct access to the objects. The relation between subject and object is mediated by artefacts (Werstch, 1991); things constructed by individuals and maintaining a dialectic relation between people and activity (Werstch, 1991). To say that a tool or artefact is mediator of learning means that it gives power to the process of transformation of objects; that it is a tool with which people think (Piteira, 2000).

This paper claims that robots can be artefacts, mediators of the learning of functions. The veracity of this claim is demonstrated in the following sections.

METHODOLOGY

The work reported in this paper was organised into three stages:

First stage – analysis of School Mathematics and Informatics curriculum; selection of didactical units where robotics can be used; creation of problems/tasks to be solved in Mathematics and Informatics classes.

Second stage –implementation of problems/tasks in Mathematics and Informatics classes; data collection through video recordings of students.

Third stage – analyses of student activity during learning with robots using interpretative methods introduced in Situated Learning Theories and Activity Theory. The unit of analysis was "(...) the activity of persons-acting in setting" (Lave, 1988, p.177).

LEARNING AS PARTICIPATION: ANALYSING STUDENTS MATHEMATICAL ACTIVITY WHEN USING ROBOTS

A brief description of mathematics class

In mathematics classes students worked in small groups. In the initial phase, the work involved construction of the robots and basic programming to solve simple tasks. This activity took place on a Windows® desktop environment and the students used a visual programming tool that ships with the robot kits. Subsequently, students used the robots to recognise and apply concepts in coordinate systems, to understand the meaning of function, to represent one function (proportionality) using an analytic expression and to intuitively relate a straight line slope with the proportionality constant, in functions such as $x \mapsto kx$.

General plan of work for functions unit

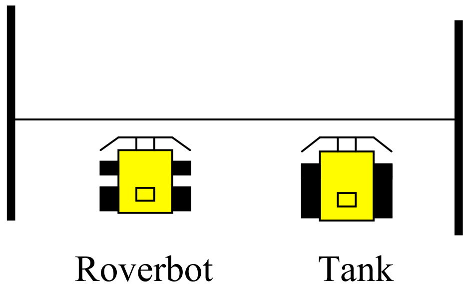
The first mathematical unit students worked on involved functions. Four sets of problems were prepared. **Problem set 1** presented examples and counterexamples of functions explaining things that take place in everyday situations. **Problem set 2** showed more complex graphs (beyond straight lines) and taught students to also recognize then as functions. In **problem set 3** it was intended that students learn proportionality as a function. The definition of proportionality emerged from the mathematical activity of students as they used robots. Finally, **problem set 4** was concerned with affin functions, such as y-intersect and slope. It also dealt with the relation between the graphical appearance of these kinds of function those of proportionality shown earlier. This paper[10] analyses students solving problem 3.

In the classroom

We will describe and analyse mathematical activity of two groups of students. One group consisted of four girls with similar mathematical levels and abilities (**C**, **La**, **Li** and **S**). When they started to work together, they had experienced considerable difficulty, even going so far as to repeatedly suggest that the problem could not be solved, at least individually. Eventually, they understood the problem could be solved if they teamed up and learned to work cooperatively. The other group featured three boys (**M**, **P** and **Ma**), in which one of them had a higher level of mathematical ability than the other two.

The class started with the teacher distributing materials to each group: one robot (either *Roverbot* or *Tank*), one laptop, one tape-measure and a worksheet including the following tasks:

I. Let's compare the two robots speed: Roverbot and Tank. Probably the first idea that occurs to you is to hold a robot race, to find out which is the quickest. However, that is not the best way to determine speed values and compare them accurately.



a) Through experimentation of Roverbot (programming, test and registration of data) complete the following table:

Time(seconds)	1	3	6
Distance covered (cm)			

(i) Calculate the quotient between distance covered and time. (ii) Do the values 'distance covered' and 'time' vary in direct proportion? Justify your answer. (iii) Which is the proportionality constant? In this situation what does the proportionality constant mean? (iv) Comment upon the following affirmation: "The correspondence between the distance covered by Roverbot and the time spent to cover that distance is a function."

Practice as meaning

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[10] For a more general discussion about mathematical activity of students using robots to learn functions see Fernandes, Fermé and Oliveira (2006, 2007, 2008)

According to Engeström (1999), in the structure of an activity we can identify subjects that act over objects, in a process of reciprocal transformations that culminates with the achievement of certain results.

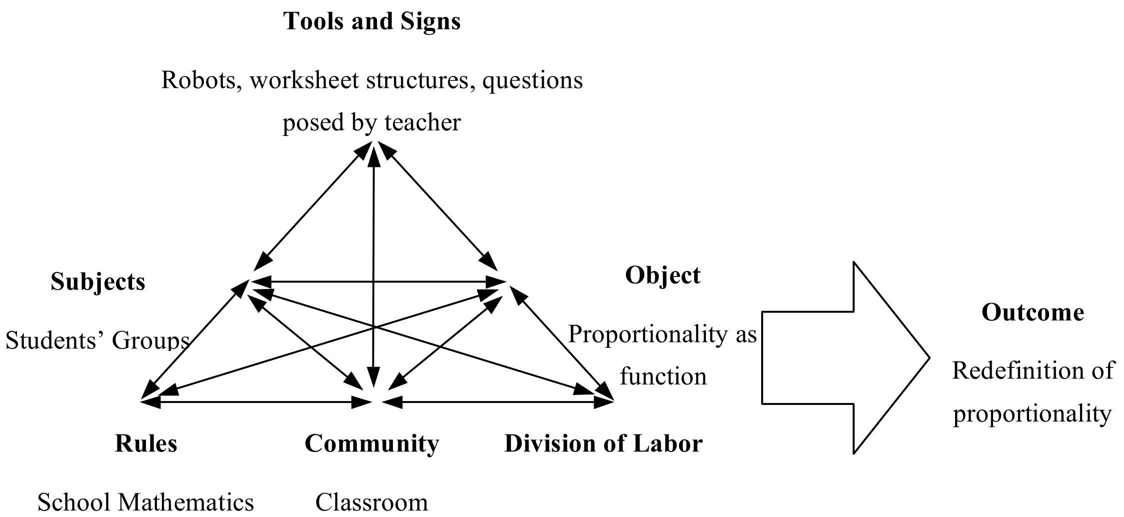
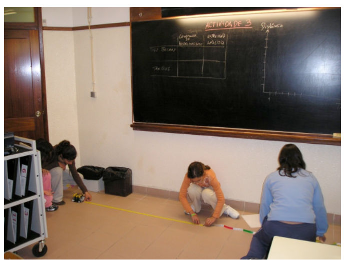


Figure 1 – School mathematics activity structure

Figure 1 shows activity during school mathematics class when robots were used to study proportionality as a function. In this case the term *subject* (figure 1) is collective and is represented by the different groups of students. The *community* is the class and its work methodology. The *object* is the 'raw material' at which the activity is directed and which is transformed (with the help of mediating instruments) as its outcome. In the situation considered here, the object is proportionality as a function and the instruments were the robots, the worksheet structure and the way the teacher posed questions to students. The episode described below shows how one of the groups solved the task described above.

Each student read the task. **C** programmed the Roverbot to move forward one second, then measured the distance covered. 33cm was recorded in the table. **S** followed the some process for 3 seconds and they registered 99cm. Then **C** programmed the robot to move forward 6 seconds. However, the desk on which they were working was too short for this last course. **Li** suggested they try out on the floor. This was done and 178 cm was recorded in the table. The students then began to discuss the results for the first time. They started to calculate the quotient between space covered and time, more or less the first times they speak. There dialogue is shown below:

1. C: 33/1 = 33 [data recorded on the worksheet].
2. C: 99/3 = 33



3. Li: 178:6 = 29.6666

4. S: It can't be. It has to be 33.

5. C: Let's programme and measure all again. Something is wrong. [They repeat all the process and the values were again 33, 99 and 178].

6. S: But it can't be. It has to be 33 (referring to the value of the quotient between the two variables)

7. La: 33 x 6 is 198. Let's put 198 on the table.

They erased 178 on the table and wrote 198. Teacher came near to the group and saw 198 (but he had previously seen 178).

8. Teacher: Wasn't the result of measuring 178?

9. C: Yes, but 33/1 is 33, 99/3 is 33

10. La: So we changed 178 by 198 because 33 times 6 is 198.

11. S: Let's programme and measure all again.

Meanwhile another group calls teacher. They programmed again the robot to forward one second and then they measured the distance covered over the desk.

12. La: Oh! I know… We measured in two different places. We have to measure always on the floor.

The results obtained of measuring the distance covered were 30, 89 and 178 for 1, 3 and 6 seconds respectively.

13. The results of the quotient were 30, 29,(6) and 29,(6) respectively. Students accepted them as good and answered that time and distant covered are in direct proportion.

Wenger (1998) states that "meaning is a way of talking about our (changing) ability - individually or collectively – to experience our life and the world as meaningful" (p. 5). He describes meaning as a learning experience.

The concept of proportionality is studied in mathematics class from 5th grade onwards. It refers to a constant relationship between two variables and is usually discussed abstractly, such as in the example below:

Verify that there is no proportionality between the following variables.

a	13	26	39	52.08
b	1	2	3	4

Many times, in school mathematics, proportionality is discussed without context; only numbers matter and the emphasis is on the mathematical concept instead of in the meaning of mathematical concept. This process makes difficult the learning experience in Wenger (1998) sense.

In the episode presented above, the students believe that the variables time and distance should be in proportion. Analysing the episode we can not determine the origin of that belief. But we can conjecture that it comes from the presence of the robot (a car) or from the way the question is written in the worksheet (question iii). Although we are guessing at its source, it is clear that the idea of proportionality is meaningful for the students, as they choose to recapture their data several times in the face of results that violate this principle. Only when an inconsistency appears, do the students begin to discuss where they made a mistake and what to do in order to solve it. But the idea that time and distance should be in proportion is really meaningful for them. This can be seen when they changed the result (from 178 to 198) to ensure that the calculations adhere to the rule and neglecting the fact of the last quotients are not equal. In spite of the evidence of the measurements, students believed that values should be in proportion. This shows that the 'dogmatic' knowledge of direct proportionality is more *entrenched*[11] than their confidence in their ability to successfully run experiments and, consequently, they neglected the evidence of the experiment.

The use of unusual artefacts in mathematics class (tape-measure, robots, laptops) associated to a methodology of work where students can stand up, measure, program the computer and experiment with data helped students to construct and rebuild meaning about the concept of proportionality.

From the perspective of activity theory, students groups acted on robots, which were mediators' elements, between them and the object. The robots were a facilitator of activity that they empowered students during the process of object transformation.

In the second student group, students had a different experience. After programming the robot for 6 seconds they had the following discussion:

> M: It's 172 cm [referring to the space covered by the robot in 6 seconds].
>
> P: 172?
>
> M: 172 or 173.
>
> P: But it can't be. It's not correct. It should be 180. And the other value should be 90 [referring to the space covered by the robot in 3 seconds].

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[11] The term entrenchment refers to Goodman (1954). He claims that the criterion to decide between two predicates (in our case, the rule and the evidence) is the degree of entrenchment of the predicates. The entrenchment of a predicate depends of the history of the past projections and their success or failure. In our case, the students have more history records where they must leave their proper ideas when confronted with the formal concepts (teacher knowledge, textbook).

Ma: Why?

P: I have done it in the calculator. If in one second the robot covers 30cm, I multiplied it by 3 and it's 90. And for 6 seconds it is 180.

M: But it's not correct. Aren't you seeing the tape measure? It's 173cm.

In this dialogue we can notice that one of the students of the group knows the scholarly notion of proportionality well and applies it to compare with the results of the experiment. He seems to trust more in the mathematical rules that he knows than in the evidence of the measurement experiment.

The two students groups reacted differently to the inconsistency between mathematical rules and the empirical evidence: one believed the values they obtained through measurement and considered that the values they obtained by approximation from the quotient were enough to guaranty the proportionality (as shown the episode above); the others calculated values after they knew the space covered by the robot in one second. Where does this difference in attitude (in the face of the same evidence) come from?

The division of labour (figure 1) refers both to the horizontal division, of the tasks between different members of the community, and the vertical, of power and status. The vertical division of labour is connected with the fact that, in the groups, there are students with more power than others (due to their superior performance in mathematics class, assessed through evaluation by their co-students) and these lead the search to solve the problem. Therefore, by analysing the horizontal division of labour we can say that it has emerged naturally between different students of the groups and represents the way how they organized their work in order to solve the problem proposed by teacher.

Finally the *rules* (figure 1) refer to the explicit or implicit regulation, to norms and conventions that constrain actions and interactions in the activity system. What students believe to be mathematics class, the way they see mathematical rules, the way they interpret the question put by the teacher and the worksheet structure (that is connected with the way they see mathematics class and mathematics) impose a certain form to the students' actions. As we have said before we have two different reactions to the inconsistency between correctness of mathematical rules and the inexactness of the empirical evidence – for one group the rules won and for other the empirical evidence.

FINAL CONSIDERATIONS

Robots helped students to renegotiate the meaning of proportionality that they had previously encountered (during seven years of school mathematics) as depending uniquely and exclusively of the quotient between two variables. The negotiation of the meaning evolves through the interaction of two process – participation and reification (Wenger, 1998). When concepts are presented to students as reified objects participation (in Wenger's sense) becomes difficult. Learning through experience, essentially negotiating meaning through participation, helps students' better grasp mathematical concepts. Most of the students in the study described here redefined the concept of proportionality as a function directly because of the work done in this mathematics class and the robots had an important role in this process (Fernandes, Fermé and Oliveira, 2006, 2007, 2008). Furthermore, as this result was not explicitly intended. Instead, it was an emergent aspect of the students' mathematical practice and study of functions. In the course of their experience with robots, students transitioned from the abstract perfection of mathematics (the definition of proportionality in school mathematics) to the practical reality (proportionality in action) of everyday experience.

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USING THE CINDERELLA/VISAGE FRAMEWORK FOR STUDYING GRAPH ALGORITHMS*

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Discrete Mathematics is a handsome pool for teaching authentic Mathematics. Using the Visage *extension of the interactive geometry software* Cinderella *– combined with the built-in programming language* CindyScript *– a simple tool for creating valuable teaching material on these topics is available. This offers a creative experimental environment not only to study theoretical properties of graphs but also to investigate in algorithms. First teaching experiences showed, how this framework inspired students at the secondary school to discover interesting results also for problems they formulated by themselves.*

Keywords: Graph algorithms, discrete mathematics, combinatorial optimization, minimum spanning tree, educational software.

INTRODUCTION

Discrete Mathematics is a modern area of mathematics coming from real life. Questions arising from combinatorial optimization problems become important e.g. in planning of railroad systems, telecommunication networks, vehicle routing and route guidance. Those questions are easy to grasp also for primary school students. On the other hand, finding optimal solutions needs a deeper insight into the mathematical structures behind the problem setting. This lets combinatorial optimization problems be an eligible starting point for an exploratory learning of mathematics.

As in other countries before (see e.g. Kenny & Hirsch 1991, Reichel & Kubelik 2002) the topic combinatorial optimization was included lately in the curriculum of the state of Berlin (LISUM 2006). Geschke et al. 2005 gave a detailed discussion of reasons to support discrete mathematics at school.

The Berlin curriculum explicitly states to use adequate computer software for teaching theses topics. In this article we want to show exemplarily how to use the interactive geometry software *Cinderella* to teach a unit on the minimum spanning

* Supported by the DFG Research Center MATHEON.

tree problem. After an introduction to the software we will report on our first experiences from a project class at a Berlin high school.

THE CINDERELLA/VISAGE FRAMEWORK

Since combinatorial graphs are usually visualized by geometric points and segments, geometry software presents itself to handle graphs on a computer. The educational software package Visage, which was developed at the DFG Research Center MATHEON, is an extension of the interactive geometry software Cinderella by Kortenkamp & Richter-Gebert (1999). The package can be used in three different settings: First, Visage is an interactive graph laboratory. The students can experiment with arbitrary graphs and standard graph algorithms. Second, Visage offers a programming interface for the implementation of further textbook algorithm or individual algorithmic ideas. Own algorithms can be implemented in Java using the Visage Java API or by using the integrated programming language CindyScript. Finally, combining the programming interface with Cinderella's ability to export any geometric construction as an interactive web applet, Visage can be used as an authoring tool for the creation of electronic worksheets. All three aspects of Visage are presented in detail in Fest & Kortenkamp (2008a). In Fest & Kortenkamp (2008b) the necessity of and the opportunities opened by connecting a modern geometry software with a programming interface are discussed.

While the Java API of Visage is meant for the development of algorithms on a higher level, we prefer the CindyScript programming interface for teaching purposes at secondary schools. CindyScript is an easy-to-learn functional language for mathematical calculations inside the geometric environment of Cinderella. The syntax of the language is very close to the mathematical language. That allows an easy translation of mathematical expressions into programming code. For example, the definition of a function like $f(x)=\begin{cases}\cos(x) & x\geq 0\\ 1-x^2 & x<0\end{cases}$ would be translated as

```
f(x):=if(x>=0, cos(x), 1-x^2);
```

With the additional statement

```
plot(f(x));
```

the function would be drawn inside the construction canvas.

CindyScript supports a direct interaction with each geometric object in the construction plane. This allows the user to get an immediate visual feedback on his implemented programming code. A very important feature is, that the source code of all existing examples and implemented algorithms can be seen and changed directly inside a resulting construction file. Having a simple access to exemplary samples stimulates the creativity and the exploratory spirit of the students enormously as we will see in section 6.

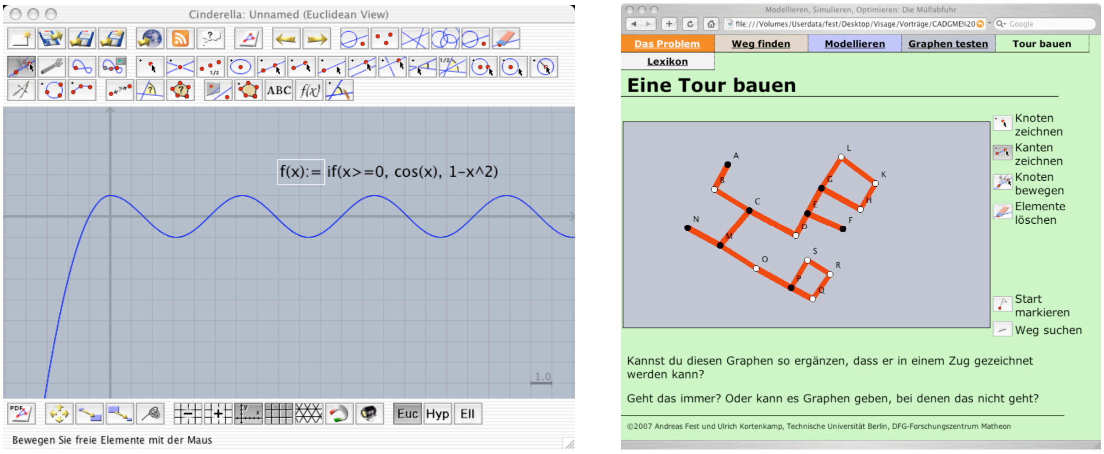

Figure 1: Using CindyScript for visualization: a) drawing a function by a simple script; b) an electronic worksheet on Eulerian graphs. Nodes of the graph are differently coloured depending on the parity of the node's degree.

Cinderella offers a variety of links between the geometric construction plane and the user's scripting code. This allows different possibilities to react on user input and to create meaningful visualizations and animations. For example, text labels can be combined with CindyScript expressions for a conditioned output of messages. Such labels also can be used as buttons releasing any user-defined action when pressed. For more complex applications Cinderella offers a script editor where self-written modules can be assigned to a bundle of different action events like mouse drags and mouse clicks, pressing of a key, redrawing the construction canvas or a time step of Cinderella' s simulation engine.

For the Visage project we developed a library of additional CindyScript functions for accessing typical graph attributes and executing and visualizing graph algorithms. We also implemented exemplary textbook algorithms on graphs to demonstrate the possibilities of our approach. Based on this work, we designed some electronic worksheets and two complete learning units on the Shortest Path Problem and on the Eulerian Tour Problem (see figure 1). A detailed description of one of the units and its didactical principals can be found in Kortenkamp (2005). Some first experiences in using the software are reported in Geschke et al. (2005).

AN ELECTRONIC WORKSHEET ON THE MINIMUM SPANNING TREE PROBLEM

We exemplarily present an electronic worksheet on the Minimum Spanning Tree Problem (MST) on graphs. In the Berlin curriculum (LISUM 2006) it is proposed to treat MST as part of the module "Diskrete Strukturen in der Umwelt" ("Discrete structures in the environment") for secondary schools at level 7/8.

The problem setting is as follows: A graph is given by nodes and edges. Each edge connects two nodes and has assigned a weight, which is a positive integer. It is asked for a spanning tree of minimum total weight, i.e. we are looking for cycle free selection of edges of the graph connecting all edges such that the sum of all edge weights is as low as possible. An exemplary graph and its minimum spanning tree are shown in Figure 2.

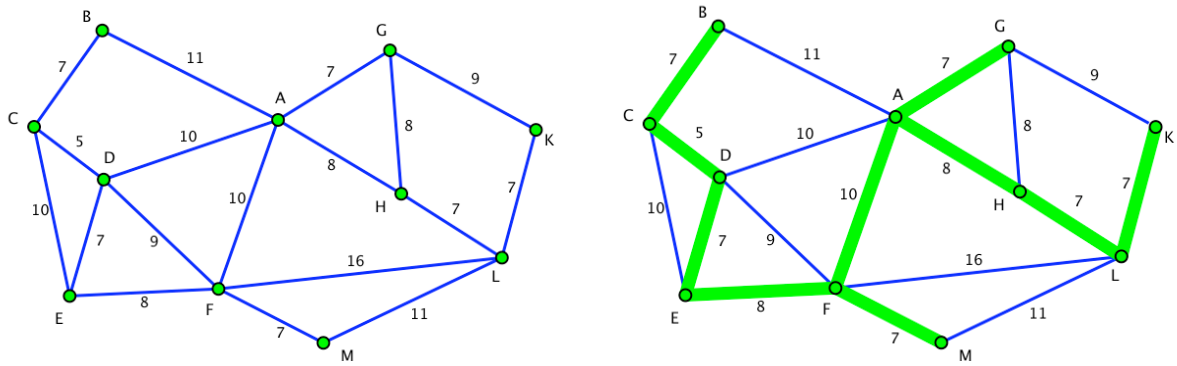

Figure 2: A weighted graph and its minimum spanning tree.

For the electronic worksheet the problem is embedded in the following background story:

> ***The caliphate Sandyrealm*** *– The caliph of the dessert country Sandyrealm wants to connect the eleven cities of his country by new roads. Every city should be reachable from each other. The new roads must be built along the already existing old caravan trails, because only there are enough oasis and wells.*
>
> *Since Sandyrealm is a poor country the road construction should be as cheap as possible.*
>
> *Which caravan trails should be developed? Which criteria do you choose? How many roads must be built?*

The problem description is supplemented by the map of the caravan trails (see fig. 3). As a metacognitive representation tool (Lajoie & Nakamura 2005) the map of Sandyrealm was implemented in an interactive worksheet. Following the guidelines of Kortenkamp (2005), we developed an interactive Cinderella construction which is embedded into a HTML document as a Java applet. The user can select or deselect any caravan path by a simple mouse click. As an immediate feedback the total length of the selected roads is displayed on the top of the map. This helps the students to avoid mistakes arising from wrong summation. An integrated solution checker can verify the user's solution. By pressing the check button the selected road network will be tested for correctness and optimality. First, it will be checked if all cities are connected, second, if the solution is cycle free (which is a necessary condition for optimality), and third, if the length of the selected road network is optimal.

The results of the solution checker are given as follows. If the selected roads are not connecting all cities, then two non-connected cities are mentioned exemplarily: "There is no connection from A to B." Otherwise, if the student's solution contains at least one cycle, he will see the message: "Can you build a shorter road net by omitting some edges?" Here, the student does not get a direct hint where to find a cycle. If the student constructed a non-optimal spanning tree, the verifier answers: "Your road network is very well! Nevertheless, can you build a shorter network?"

And finally, when the student found a minimum spanning tree, he gets an affirmation: “Fine! You have found an optimal road net!”

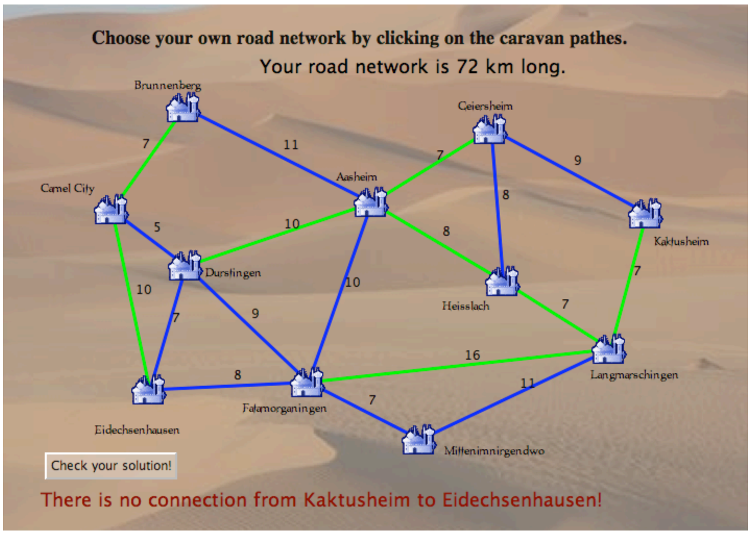

Figure 3: The electronic worksheet for MST. The user may select/deselect edges. The total length of the chosen road network is displayed. After pressing the checker button, the user’s solution is verified and a corresponding hint is displayed.

The whole solution checker is implemented independently from the instance, i.e. it will work correctly on any arbitrary weighted graph. As a consequence, students can load the worksheet into Cinderella and change the graph or construct a new one without loosing the correctness of the solution verifier. That works since the right solution is not directly saved in the worksheet but is recalculated each time the check button is pressed, i.e. we implemented an algorithmic verifier. The test for connectivity and cycle freeness is done by a breadth first search on the set of selected edges. If this test successfully identifies a spanning tree, its total weight is compared with the weight of an optimal solution that was found by passing once Prim’s MST algorithm.

TEACHING MINIMUM SPANNING TREES

In December 2007 a project week with the topic “Spannungen” [1] took place at the Luise-Henriette-School in Berlin. Nine students of all class levels from the secondary school explored real life problem settings with the methods of the algorithmic graph theory during a three-day project class “Günstig verbunden – Minimale aufspannende Bäume”, which was inspired by Lutz-Westphal (2007a). We used the electronic worksheet mentioned in the previous chapter as a starting point for a teaching unit on the MST problem. The students developed their own solution strategies with and without the computer, even far beyond the initial problem setting. They used different methods to present the discovered algorithms: posters, flipbooks, role playing games, and, finally, programming with Visage.

The project class was held in an open learning environment and was structured into three phases: Concept formation phase, Development phase, and Presentation phase.

Concept formation phase: The initial exercise was presented with a paper version of the electronic worksheet. The students additionally got multiple sheets of the map for the development of their own solution. A particular aim in this phase was to reflect on the structure of the solutions developed by the students. During their first attempts the students only used one of the sheets, so after a while they had set as many marks and labels on the paper map that they could no more see any structure in their solution trials. After an adequate handling time they got the electronic version of the worksheet. Due to the correctness checker of this tool the students were able to create good solutions and to discover the structure of their approach. Now, they had the capacity to verbalize their own way of finding a solution during the group discussion. Most of the students' solutions were based on Kruskal's algorithm, since it seemed to be natural to the students to choose always the cheapest edge. Finally some terms like *graph* and *tree* were acquired by discussion.

Development phase: The main part of the project was used for the development of algorithmic ideas and presentation forms for those. The students could select activities from a list, or they attended to self-developed activities. They worked in small, varying groups, which they composed on their own. Some chosen activities are described below.

Presentation phase: The project days were finished by a school wide exhibition for all schoolmates, teachers and parents. Each project had to present its results. All project members had to be able to explain at least the main ideas of any result developed by the project class.

The following list is a selection of the students' activities during the development phase.

Graphs and trees: Basic properties of graphs and trees were elaborated and proved. Analogies between trees in graph theory and in nature were worked out. The results were presented on a poster.

Investigation on applications: The students read up on real life applications of minimum spanning trees. They used literature and the world wide web as a source. Interesting applications were depicted on a poster.

Understanding and visualizing algorithms: Based on the ideas worked out in the concept formation phase, own solution algorithms for the MST were developed and compared with standard textbook algorithms. It turned out that all solutions of the students were based on the ideas of the algorithms by Kruskal and by Prim (see e.g. Korte & Vygen 2001). The principles of the algorithms were carried out and displayed on another poster.

Creating flipbooks: Another way to visualize the flow of an algorithm is to create a flipbook (Lutz-Westphal 2007a). Each algorithmic step is displayed in a new page of the book. To create a correct flipbook it requires a deeper understanding of the main ideas of the algorithm.

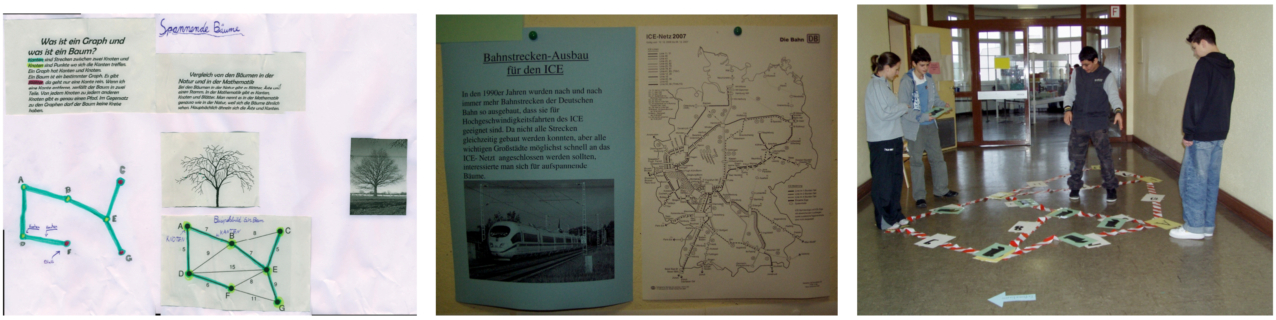

Figure 4: Different activities of the development phase. a) Poster: Graphs and trees. b) Poster: German railroad as an application. c) Performing a role playing game.

Designing and performing a role playing game: A huge graph was constructed on the floor by using coloured paper and red barrier tape. Simple standard graph algorithms were reformulated as an instruction to a role playing game which was executed on the floor graph. Because of the large dimension of the floor graph one cannot survey the whole graph from outside. This simulates the constrained access that a computer has on the data structures of the graph and forces the students to act according to the algorithmic steps just as it is written in the algorithmic flow.

Implementing the algorithms and programming visualizations: Self-developed and textbook algorithms were implemented using the CindyScript programming language. As a basis the students got a worksheet with the exemplary implementation of the visualization of one standard algorithm. They adopted the programming code to visualize some other algorithms. Fest and Kortenkamp (2008a) reported on a similar approach that was previously used successfully for a university course for teacher students. This course was held at the TU Berlin by Lutz-Westphal (2007b).

Variation of the problem setting: The students varied the given problem and formulated different questions to be discussed. They studied alternative combinatorial optimization problems and developed and implemented self-designed software for those questions.

…AND LEARNING BEYOND

Motivated by their success in programming a simple algorithm a group of three older students decided to solve a different optimization problem on graphs by programming. One of the students already learned about algorithms for calculating shortest paths in graphs like roads or railway networks. The idea was to implement an interactive route guidance system for the Berlin subway similar to the official “Fahrplanauskunft” on http://www.fahrinfo-berlin.de/. The students took a clipping of the subway route network and imported it as a background image into a Cinderella construction. Then they modelled the stations and track sections as nodes and edges of a graph. Finally, they implemented the breadth first search algorithm to calculate shortest paths in their graph and added a user interface where the user can select an initial station and a destination for his request. The calculated shortest path is then displayed immediately on the line map as depicted in figure 5a).

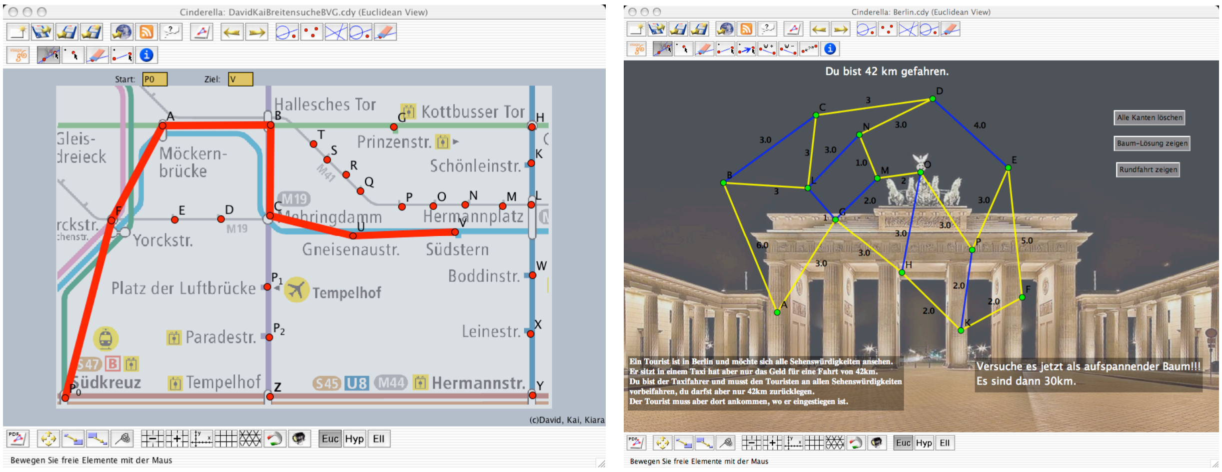

Figure 5: Two electronic worksheets developed by students. a) An interactive route finder for the Berlin subway net. b) A worksheet on the Travelling Salesman Problem.

Leuders (2005) reported on some technical difficulties to map a real life road net into the data structures for the graph during a different teaching project held on “Route guidance”. His project class used algorithms for image processing to extract the structure of the graph from a scanned road map. By using Visage our students avoided these problems because now they had a simple tool for modelling the graph by hand. The software provides the corresponding data structures for the graph automatically.

An even more surprising and interesting result was created by a group of three students of the eighth degree. Inspired by the initial electronic worksheet, they began to create similar worksheets with varying exercises. Although they had no programming skills and never used the software Cinderella before, they managed to implement their ideas for a user interface and for visualizations. They required only a short introduction by the teacher to modify and adapt the initial worksheet. Even complete solution algorithms were taken from other exemplary Cinderella constructions and integrated into their own work.

The students decided to draw a graph depicting a simplified city map. Then they formulated the exercise to plan a sight seeing tour driven by a taxi through the whole city. The resulting worksheet is shown in figure 5 b). The students’ first approach for the solution was to create – again – a minimum spanning tree. After a teacher’s question whether this solution is a practicable tour for the taxi, the students argued that a taxi must use each edge of a tree solution twice. They observed that they can find a better solution by searching short cuts. Now, they constructed an optimal tour by hand and – since they didn’t know any solution algorithm – coded this solution directly into the worksheet. In fact, the formulated problem is an instance of the Travelling Salesman Problem (TSP), which is a **NP**-hard optimization problem. Until now, no efficient solution algorithm for this problem is known. But the discussion of their first approach drove the students finally to a rediscovery of the Double-Tree approximation algorithms for the metric TSP (see e.g. Korte & Vygen 2001). This result is usually taught to college students in specialised courses.

CONCLUSIONS AND FUTURE WORK

Due to the special settings of the project class, we cannot yet give established results on using the unit in everyday teaching. Nevertheless, the experiences in the project class, based on observations of the students' activities, already showed that the use of the Cinderella/Visage framework in an open learning environment leads the students to creative results.

The use of the electronic worksheet based on the teaching software Visage seemed to be very helpful for the students to structure their own thoughts. The impact of the students' interaction with the electronic worksheet must be further investigated in an empirical study. By all means, the worksheet motivated the students to follow own ideas far beyond the initial problem setting and to create new electronic worksheets.

Programming is an inherent part of working on combinatorial optimization problems. Hence the students get an authentic feeling on working in this area. Due to the easy-to-learn programming language CindyScript and its integration into the geometry software Cinderella even the students of the eighth degree were able to create own electronic worksheets without prior knowledge of computer programming. The students developed basic programming skills as a spin-off product from designing their own worksheet. The older students managed to implement the mentioned algorithms without any problems in a surprisingly short time. A visualization of Kruskal's algorithm was finished in less than two and a half hours. Here again the used programming language was learned on the fly by applying it during the implementation. Despite the simplicity of the CindyScript programming language the teacher is forced to gain substantiated knowledge of the programming language himself, once the software Cinderella/Visage is used by the students for implementing their own solutions. Otherwise he will not be able to support the students adequately.

Since implementing the considered algorithms requires a repeated mental structuring of their knowledge, it might help the students to deepen their understanding of the algorithms. This, again, must be examined in more detail in a further study.

NOTES AND ACKNOWLEDGEMENTS

1. The german word "Spannung" means "friction" and "strain" as well as "tension", "pressure" or "voltage". In the figurative sense the meaning of "spanning" in "spanning tree" also fits to the subject of the project days.

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A LEARNING ENVIRONMENT TO SUPPORT MATHEMATICAL GENERALISATION IN THE CLASSROOM

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This paper discusses classroom dynamics and pedagogical strategies that support teaching mathematical generalisation through activities embedding a specially-designed microworld. A prototype of our microworld was used during several one-to-one and classroom studies. The preliminary analysis of the data have allowed us to see the implications of designing and evaluating this specific technological tool in the classroom as well as the teachers' and the students' requirements. These studies feed into the design of the intelligent support that we envisage the system will be able to offer to all students and the teacher. In particular, they helped us identify which aspects of teachers' interventions could be delegated to our system and what types of information would be useful for supporting teachers.

Keywords: Mathematical Generalisation, Microworlds, Classroom Practices, Teachers, Intelligent Support

INTRODUCTION

It seems that there is a growing diversity of computer-assisted material and tools for mathematics classrooms. Even though this proliferation of digital tools and new technologies has broadened the instructional material available for teachers, they are still rather insignificant to classroom practice and their use is far from regular (Artigue, 2002, Mullis et al., 2004, Ruthven, 2008). This suggests a challenge for mathematics educators to develop complete, consistent and coherent systems that not only assist students, but also support teachers' practice in the classroom.

The aim of the MiGen[1] project is to design and implement a system with teachers that meets their as well as students' requirements. We are developing an intelligent exploratory learning environment for supporting students in making mathematical generalisations. In more detail, our focus has been on the difficulties, first students face in their efforts to generalise and second teachers face in their efforts to support students appropriately during lessons with 20-30 students. For our initial investigations, we restricted the domain of mathematical generalisation to the generation and analysis of patterns. Activities with patterns often appear in the UK mathematics curriculum and have been identified as motivating for students (see Moss & Beatty, 2006). They also comprise a good domain for generalisation, since they allow students to come up with different constructions for the same pattern, find the corresponding rules and realise their equivalence.

Our aim is to develop a system that provides the means to understand the idea of generalisation, but also the vocabulary to express it, while supporting rather than supplementing the teacher. The system is intended to provide feedback to the teacher about their students' progress and, where the system's 'intelligence' is unable to help students, to prioritise the students in critical need of the teacher's assistance.

The core of our system[2] is a microworld, called the eXpresser (described briefly in the next section), in which students can construct and analyse general patterns using a carefully designed interface. In order to build the microworld, our team[3] started with a first prototype (Pearce et. al, 2008). Using an iterative design process, and in order to investigate the effectiveness of our approach, we carried out a number of studies with individual students or pairs of students, each time using the feedback we obtained to build the next prototype. This process resulted in the evolution of the prototype and its subsequent evaluation in classroom.

This paper, after a brief discussion of our methodology, presents the preliminary data analysis of the classroom studies that not only support the next version of the microworld, but also feed into the design of the intelligent support that we envisage the system will be able to provide. Our focus here is on the teachers' pedagogical strategies and the students' needs for support and assistance during their interactions with the microworld. This analysis is followed by a discussion of the teachers' interventions that could be delegated to the 'intelligent' system and what types of information would be useful for supporting teachers and therefore necessary for the development of the intelligent support components of our and other similar systems.

A microworld for patterns – the eXpresser

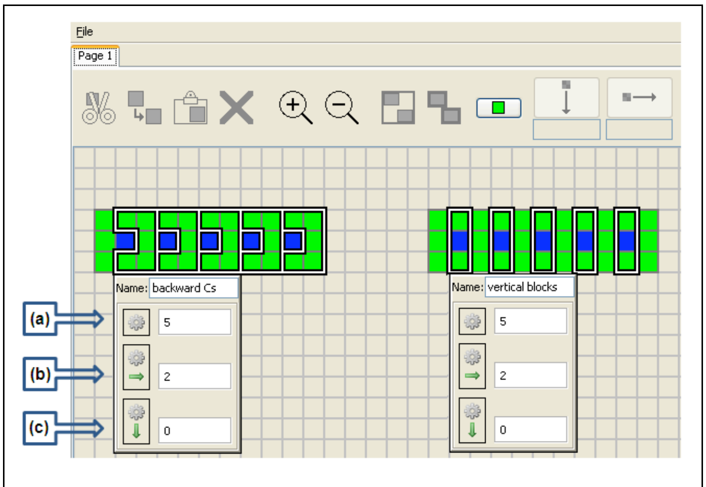

Figure 1. The interface of eXpresser with two different constructions of the same pattern. The left one is made out of a vertical block of 3 squares and 5 'backward C-s' and the right one of alternating vertical blocks.

First, we present briefly the main features of the eXpresser. We emphasise that at the stage of the study, attention was focused largely on the features key to our research goals. So, the following description of the system is by no means complete. In addition, its design has evolved significantly through studies such as the ones described in this paper. The interested reader is referred to Noss et al. (2008), where the system's rationale and design principles are described in detail.

In eXpresser, students can construct patterns based on a 'unit of repetition' that consists of square tiles. These patterns can be combined to form complex patterns, i.e. a group of patterns. A pattern's property box (depicted in Figure 1) shows three numeric attributes that characterise the pattern[4]. The first specifies the *element count* (number of repetitions) of this pattern (a). The icon with the right arrow (b) specifies *how far to the right* each shape should be from its predecessor and, similarly, the icon with the down arrow (c) specifies *how far down* a shape should be.

A requirement of our constructivist approach was to allow students to construct patterns in a variety of ways (Figure 1). Additionally, an important design feature is the ability to 'build with n' (see Noss et al., 2008), i.e. to use independent variables of the task to create relationships between patterns.

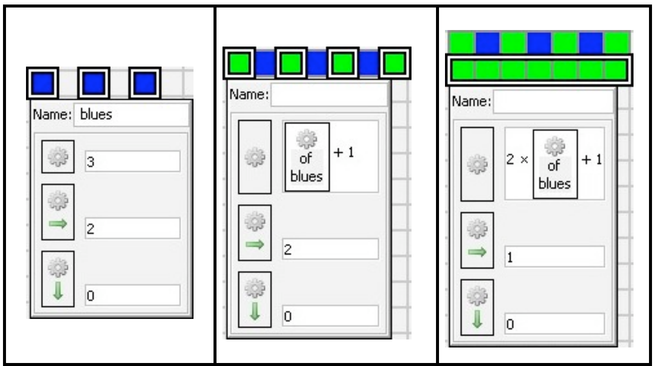

Figure 2. Another way to construct the pattern in Figure 1. To relate the middle row with the first pattern (named "blues"), the number of repetitions should be one more than the number of repetitions of "blues". For the bottom row it should be twice more plus one. These relationships are specified iconically.

This feature not only provides students additional ways to construct patterns but we hypothesised that it enables students to realise what are the independent variables and use them to express relationships. To overcome difficulties that students face with symbolic variables the microworld employs what we call 'icon-variables', which are pictorial representations of an attribute of their construction. We have illustrated in previous work (Geraniou et al., 2008), that these 'icon-variables' provide a way to identify a general concept that is easier for young learners to comprehend. An example of expressing such relationships is depicted in Figure 2.

METHODOLOGY

Our own previous work and studies by Underwood et al. (1996) and Pelgrum (2001), for example, concerning the adoption of educational software in classrooms emphasise the importance of teachers' involvement in the whole design process of computer-based environments. Therefore, several meetings with the teacher were held before each classroom session so that they were familiarised with the prototype, agreed and made input to the lesson plans and in order to clearly state the teacher's, the students' as well as the researchers' objectives.

The overall methodological approach is that of 'design experiment', as described by Cobb et al. (2003). One of our goals during these sessions was to inform our system's design and evaluate the effectiveness of our pedagogical and technical approach. We aimed at investigating the classroom dynamics by looking at individual students' interactions with the microworld, the collaboration among pairs or groups of students as well as the teachers and researchers' intervention strategies.

We investigated the use of eXpresser in several one-to-one and classroom sessions with year 7 students (aged 11-12 years old). Particularly for the classroom sessions, two researchers played the role of teaching assistants and another was observing and keeping detailed notes regarding the researchers' and the teacher's interventions. The sessions were recorded on video and later analysed and annotated with the help of the written observations. Based on these, we were able to get information regarding the time and duration of the interventions, the type of feedback given, the students' reactions and immediate progress after the interventions. Therefore, our goals in the study reported in this paper were to identify not only the students' ability to collaborate successfully and articulate the rules underpinning their generalisation of the patterns but particularly when and how the teacher or the researchers intervened.

However, to maintain the essence of exploratory learning, research suggests a teacher's role should be that of a 'technical assistant', a 'collaborator' (Heid et al., 1990), a 'competent guide' (Leron, 1985) or a 'facilitator' (Hoyles & Sutherland, 1989). Our aim was to achieve the right balance between students' autonomy and responsibility over their mathematical work and teachers' and researchers' efforts to scaffold and support their interactions. The teacher and the researchers set out to adopt this role by following a specific intervention philosophy that adhered to our framework of interventions (Mavrikis et al., 2008), which was based on our previous work with Logo and dynamic geometry environments. This framework was extended after the analysis of the data and is presented in the 'Classroom Dynamics' section. Our aim was to avoid imposing our (or the teacher's) views or ways of thinking, but instead allowing students to express their viewpoints and assist them by demonstrating the tools they could use: for example, by directing their attention, organising their working space and monitoring their work.

CLASSROOM SCENARIO

We illustrate here a classroom scenario carried out with a year 7 class with 18 high-attaining students. Students were introduced to the microworld through a familiarisation process, during which the teacher introduced all the key features to construct a simple pattern and students followed his actions on their laptops.

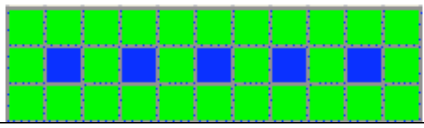

Figure 3. The activity: Find a rule for calculating the number of green (light) tiles for any chosen number of blue (dark) ones.

Students were then presented with the task in Figure 3. The pattern was shown dynamically on the whiteboard; its size changed randomly showing a different instance of the pattern each time. This made it impossible for students to count the number of tiles while allowing them to 'see' variant and invariant parts of the pattern. We hypothesised that a dynamically presented task would discourage 'pattern-spotting', which focuses on the numeric aspect of specific instances of the pattern, and counting, which encourages constructing specific cases of the pattern. It also provided a rationale for the need of a general rule that provides the number of tiles for *any* instance of the pattern.

Students were given the freedom to construct the pattern in their own way, using the system's features they had been shown earlier. They were asked to write on a hand-out how they constructed the given pattern and then discuss in pairs their constructions and the methods they followed. They also worked collaboratively to find a rule that gives the number of green tiles for any chosen number of blue ones. Students' next challenge was to find different ways to replicate the pattern and describe them on the hand-out explicitly, so as their partner could understand it._After discussing with their partner, if they had come up with the same constructions, they were expected to try to see whether there were any other ways and find all the rules that represented their constructions and write them down. Finally, the teacher initiated a discussion, where students were asked to present their rules to the rest of the class. Rich arguments were developed and students challenged each other to justify the generality of their construction and the rules they have developed.

During this classroom study many interesting issues regarding the classroom dynamics were identified that informed our further design of the microworld and the overall system and the next phase of the research.

CLASSROOM-DYNAMICS

As expected, to ensure the success and effectiveness of students' interactions with the eXpresser, there was a need for significant support from the teacher and the researchers. As discussed already, we had agreed a specific intervention philosophy with the teacher. The analysis of the data (video recordings and written observations) revealed further strategies and extended our previous framework of interventions (Mavrikis et al., 2008). The revised framework is presented in Table 1.

- Reminding students of the microworld's affordances
- Supporting processes of mathematical exploration
  - *Supporting students to work towards explicit goals*
  - *Helping students to organise their working environment*
  - *Directing students' attention*
  - *Provoking cognitive conflict*
  - *Providing additional challenges*
- Supporting collaboration
  - *Students as 'teaching assistants'*
  - *Group allocations*
  - *Encourage productive discussion (group or classroom)*
- Ensuring task-engagement and promoting motivation

Table 1. Types of interventions observed during our studies

Below we pull out some illustrative episodes under each category.

Reminding students of the microworld's affordances

As facilitators the teacher and the researchers (referred to as 'facilitators' for the rest of the paper) managed to support students' interactions and explorations by reminding them of various features of the system that assisted students' immediate goals. This intervention acted sometimes as a prompt and other times as an offer of assistance. If the facilitator sensed a student was working towards a direction where they could be assisted by a specific tool, they would point it out to their students. This teaching strategy might have proved rather common as for some students the one lesson spent on familiarisation with the system seemed not enough.

Supporting processes of mathematical exploration

We often needed to support the students' problem-solving strategies. For example, we noted that students tended to forget their overall goal. Students seemed to get lost in details and got carried away with various constructions ('drawings'), which, even though offering students more experience of the system's features and affordances, it sometimes led them in the wrong direction. One of the downsides of any microworld is that students' actions can become disconnected from the mathematical aspects under exploration. Even though, the system's affordances were carefully designed to support students' thinking processes, they were not always naturally adopted by them. Therefore, when needed, we provided a reminder of their goals or helped them re-establish them by asking questions like "What are you trying to do?" or "What will you do next?" (*supporting students' work towards explicit goals*).

Another aspect of problem-solving skills (particularly when working in microworlds) that some students seemed to lack was being able to come up with *an organised working environment*. We occasionally advised students to delete shapes that were irrelevant to the solution or change the location of a shape so that they could concentrate on ones that could prove useful. It was evident that students who worked effectively and reached their goals were the ones that organised their working space and therefore supported their perception of the task in hand.

Directing students' attention was a necessary pedagogic strategy. We prompted students to notice invariants or other details which are important for their investigations without giving away the answer. For example, we asked questions such as "Did you notice what happened when you increased the length of this pattern?" or "when you changed this property of your pattern?". These pointed out certain facts that students might have missed out or ignored, but also exposed possible misconceptions and misinterpretations. If students were focusing on or manipulating unnecessary elements of their construction, the facilitators provided hints towards more constructive aspects. If students' responses revealed any misconceptions, then such a prompt acted as an intervention for *provoking cognitive conflict*. There were cases where the cognitive conflict was not obvious to the students directly and further explanations were required from the facilitators. These normally involved giving counter-examples to provoke students' understanding and challenge their thinking processes. Besides this intervention we used another strategy, referred to as "messing-up", used in our previous work in dynamic geometry (Healy et al., 1994). This strategy challenged students to construct a pattern that is impervious to changes of values to the various parameters of the tasks. Students tended to construct patterns with specific values and had their constructions 'messed-up' when the facilitators suggested: "What happens when you change this to say 7 (a different value to the student's chosen one)?". This strategy gave a rationale for students to make their constructions general by encouraging them to think beyond the specific case. In other cases where students seemed to have reached a satisfactory general construction, the facilitators intervened by *providing additional challenges*. For example, "Could you find another way of constructing the pattern?".

Supporting collaboration

Students who achieved a seemingly general construction and found a rule (general or not, representing their construction or not), often failed to find different ways of constructing the pattern. Our approach in these circumstances was to introduce them to the collaborative aspect of the activity, in which they had to discuss, justify and defend the generality of their constructions and their rules to their partners. We envisaged that learners' general ways of thinking would be enhanced by the sharing of their different perspectives. Accompanied by the facilitators' or fellow students' assistance, students could appreciate the equivalence of their approaches and possibly adopt a more flexible way of thinking. In this study, the rationale behind collaboration was to give students an incentive to enrich their perception and understanding of the given pattern, to find more ways of constructing it and begin to appreciate their equivalence mathematically. The allocation of students to groups aimed at ensuring the best possible collaboration (*group allocations*). Ensuring though that discussions carried out within the groups were fruitful was not an easy task. The first step towards this goal was grouping the students in a way that promoted participation from all members of the group while discouraging students from dominating a discussion (*encourage productive discussion*).

On some occasions, the facilitators, particularly the teacher who has better insights into his students' competence, encouraged students to take the role of a '*teaching assistant*' and help others who were less successful in their constructions. This intervention boosted students' confidence, but also gave them an opportunity to reflect upon their actions and an incentive to explain their perspective.

Ensuring engagement and promoting motivation

Finally, although the activities and the system affordances were designed to assure engagement as well as promote students' motivation, there were various occasions (e.g. being stuck or 'playing' by drawing random shapes) when the facilitators' intervention was required. Our vision was to give the right rationale for students to solve the task and praise their efforts. These studies supported our view that avoiding tedious activities that were pointless in the students' eyes, not only reduces the risk of off-task behaviour, but also sustains a productive atmosphere for students.

TOWARDS AN INTELLIGENT SYSTEM IN THE CLASSROOM

The interventions that were discussed above require an intensive one-to-one interaction with the students who require help. However, it is unrealistic to expect teachers in classrooms to be able to adhere to the demanding role of facilitators, keeping track of all students' actions while allowing them to explore and have the freedom to choose their immediate goals. As mentioned above, there are multiple ways of constructing a pattern and therefore multiple ways of expressing general solutions for such activities. It is at this point that the value of a system that can provide information to the teacher becomes apparent.

One of the most practical issues regarding students' interactions in such environments is that despite the familiarisation process, there is a need to remind students of certain features or even prompt them to use those which could prove useful for their chosen strategy. Therefore, it should be possible to identify (based on students' actions) which tasks of the familiarisation activity they should repeat. An intelligent system could highlight tools relevant to their current actions or offer a quick demonstration directly taken from their familiarisation activity. Furthermore, it could repeat their previous successful interactions relevant to the current activity.

In terms of the teachers' responsibility to attend to and help all the students in a classroom our studies highlighted the difficulty to prioritise which student to help. It is inevitable, therefore, sometimes to offer support to students who do not need it as much as others or even leave some students unattended due to the time constraints of a lesson. Moreover, it is possible for students to misunderstand certain concepts and leave a lesson with a false sense of achievement. Of course, it is difficult for an intelligent system to detect this accurately. However, it is possible to draw the teacher's attention to students potentially in need. By providing therefore information regarding students' progress at various times during a lesson as well as alerting them of likely misconceptions, it becomes possible for the teacher to spend their time and effort efficiently.

Besides these teachers' difficulties, there are situations when, despite having carefully-planned lessons, teachers are required to take immediate and effective decisions during lessons to accommodate their students' needs. For example, noticing when students are having difficulty with certain tasks or providing extension work are interventions which could be delegated to our system, allowing more time for teachers to provide essential help. Moreover, the collaborative component of an activity could be supported by the system by recommending effective groupings of students and allowing them to co-construct patterns whilst reducing dominance and promoting successful collaboration. The system could inform the teacher about the dynamics of different groups and alert them of possible concerns regarding the groups' progress as well as suggest more productive groupings (e.g. group students with different constructions but equivalent general expressions).

In addition, although we acknowledge the strong dependency between motivation, engagement and the design of the activities, it was evident that some students were at points disengaged. Even if off-task behaviour can sometimes lead to fruitful outcomes and intrigue students' thinking processes towards a direction, there is a need in automatically detecting such behaviour and informing the teacher. It then becomes the teacher's responsibility to decide how and whether to intervene.

The aforementioned suggestions for intelligent support could ease the use of an exploratory environment like the eXpresser in the classroom. It is often the case that such systems end up being used as a tool just to demonstrate certain mathematical concepts because of similar difficulties faced in classroom as those we reported here. Moreover, although quite a few 'intelligent' tutoring systems have been designed to provide support and personalised feedback to students and are starting to be integrated in classroom (Forbus et al., 2001), they usually scaffold the students with predetermined solution methods and by definition restrict students' reaching their own generalisations. Our team's challenge is to build a system that provides students the freedom to explore, make mistakes, get immediate feedback on their actions while assisting teachers in their difficult role in the classroom and therefore enable the successful teaching and learning of the idea of mathematical generalisation.

NOTES

1. See http://www.migen.org/ for details. Funded by the TLRP, e-Learning Phase-II; Award no: RES-139-25-0381.

2. Our system comprises of two additional components, the eGeneraliser, which aims to provide students with personalised feedback and support during their interactions with the microworld, and the eCollaborator, which aims to foster an online learning community that supports teachers in offering their students constructions and analyses to view, compare, critique and build on.

3. We would like to acknowledge the rest of our research team and particularly Sergio Gutierrez, Ken Kahn and Darren Pearce who are working on the development of the MiGen system.

4. Each attribute has an associated icon tentatively depicted as cogs "to indicate the inner machinery of a pattern". As the design of eXpresser is evolving our team is evaluating the appropriateness of these icons.

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TEACHING RESOURCES AND TEACHERS' PROFESSIONAL DEVELOPMENT: TOWARDS A DOCUMENTATIONAL APPROACH OF DIDACTICS

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In this paper we propose a theoretical approach of teachers' professional development, focusing on teachers' interactions with resources, digital resources in particular. Documents, entailing resources and schemes of utilization of these resources, are developed through documentational geneses occurring along teachers' documentation work (selecting resources, adapting, combining, refining them). The study of teachers' documentation systems permits to seize the changes brought by digital resources; it also constitutes a way to capture teachers' professional change.

Keywords: Documents, Geneses, Professional development, Resources, Teachers

INTRODUCTION

We present in this paper the first elements of a theoretical approach elaborated to study teachers' development, and in particular teachers ICT integration.

The questions of technology integration, and the way teachers work in technology-rich environments, have been extensively researched, and discussed at previous CERME conferences (Drijvers *et al.*, 2005, Kynigos *et al.*, 2008). Ruthven's presentation at CERME 5 drew attention on the structuring context of the classroom practice, and on its five key features: *working environment, resource system, activity format, curriculum script, time economy* (Ruthven, 2008). This leads in particular to consider ICT as part of a wider range of available teaching resources. This view also fits technological evolutions: most of paper material is now at some point in digital format; teachers exchange digital files by e-mail, use digital textbooks, draw on resources found on websites etc. Considering ICT amongst other resources raises the question of connections between research on ICT and resources-oriented research.

Many research works address the use of *curriculum material* (Ball & Cohen, 1996; Remillard, 2005). They observe the influence of such material on the enacted curriculum, but also highlight the way teachers shape the material they draw on, introducing a vision of "curriculum use as participation with the text" (Remillard, 2005, p.121). Other authors consider more general resources involved in teaching: material and human, but also mathematical, cultural and social resources (Adler

2000). They analyze the way teachers interpret and use the available resources, and the consequences of these processes on teachers' professional evolution.

Such statements sound familiar for researchers interested in ICT, who "consider not only the ways in which digital technologies shape mathematical learning through novel infrastructures, but also how it is shaped by its incorporation into mathematical learning and teaching contexts" (Hoyles & Noss, 2008, p. 89). Conceptualization of these processes is provided by the *instrumental approach* (Guin *et al.*, 2005) and from the work of Rabardel (1995) grounding it; this theoretical frame has contributed to set many insightful results about the way students learn mathematics with ICT. Further refinements of this theory have led to take into account the role of the teacher and her intervention on students instrumental geneses, introducing the notion of *orchestration* (Trouche, 2004). Considering instrumental geneses for teachers has been proposed in the context of spreadsheets (Haspekian, 2008) and e-exercises bases (Bueno-Ravel & Gueudet, 2008). These refinements can be considered as first steps towards the introduction of concepts coming from the instrumental approach and illuminating the interactions between teachers and ICT.

Thus connections between studies about the use of teaching resources, and studies about the way in which teachers work in a technology-rich environment exist; however, elaborating a theoretical frame encompassing both perspectives requires a specific care. We present here an approach designed for this purpose, and aiming at studying teachers' documentation work: looking for resources, selecting, designing mathematical tasks, planning their order, carrying them out in class, managing the available artifacts, etc. We take into account teachers' work in class, but also their (too often neglected) work out of class.

We draw on the theoretical elements evoked above, but also on field data. Some of these data come from previous research in which we were engaged: particularly about use of e-exercises bases (Bueno-Ravel & Gueudet, 2008), and about an in-service training design, the SFoDEM (Guin & Trouche, 2005). Other data were specifically collected: we have set up a series of interviews with nine secondary school teachers. We chose teachers with different collective involvements, different institutional contexts and responsibilities, and different ICT integration degrees (Assude, 2008). We met them at their homes (where, in France and for secondary teachers, most of their documentation work takes place), and asked them about their uses of resources, and the evolution of these ways of use. We observed the organization of their workplaces at home, of their files (both paper and digital), and collected materials they designed or used. The analyses of these data contributed to shape the concepts; in this paper we only use them to display illustrations of the theory. All the interviews took place in France; thus the national context certainly influences the results we display. We hypothesize nevertheless that the concepts exposed are likely to illuminate documentation work in diverse situations.

We present in section 2 the elementary concepts of this theory, introducing in particular a distinction between *resources* and *documents*, and the notion of *documentational genesis*. This theory entails a specific view of professional evolutions; we expose this view and its outcomes in section 3.

RESOURCES, DOCUMENTS, DOCUMENTATIONAL GENESES

The instrumental approach (Rabardel, 1995, Guin *et al.*, 2005) proposes a distinction between *artifact* and *instrument.* An artifact is a cultural and social means provided by human activity, offered to mediate another human activity. An instrument comes from a process, named *instrumental genesis*, along which the subject builds a *scheme* of utilization of the artifact, for a given class of situations. A scheme*,* as Vergnaud (1998) defined it from Piaget, is an *invariant organization of activity* for a given class of situations, comprising in particular rules of action, and structured by *operational invariants,* which consist of implicit knowledge built through various contexts of utilization of the artifact. Instrumental geneses have a dual nature. On the one hand, the subject guides the way the artifact is used and, in a sense, *shapes* the artifact: this process is called *instrumentalization*. On the other hand, the affordances and constraints of the artifact influence the subject's activity: this process is called *instrumentation*. We propose here a theoretical approach of teaching resources, inspired by this instrumental approach, with distinctive features that we detail hereafter, and a specific vocabulary.

We use the term *resources* to emphasize the variety of the artifacts we consider: a textbook, software, a student's sheet, a discussion with a colleague etc. A resource is never isolated: it belongs to a set of resources. The subjects we study are teachers. A teacher draws on resources sets for her documentation work. A genesis process takes place, bearing what we call *a document*. The teacher builds schemes of utilization of a set of resources, for the same class of situations, across a variety of contexts. The formula we retain here is:

Document = Resources + Scheme of Utilization.

A document entails, in particular, operational invariants, which consist of implicit knowledge built through various contexts of utilization of the artifact, and can be inferred from the observation of invariant behaviors of the teacher for the same class of situations across different contexts.

Figure 1 represents a *documentational genesis*. The instrumentalization process conceptualizes teacher appropriating and reshaping resources, and the instrumentation process captures the influence, on the teacher's activity, of the resources she draws on.

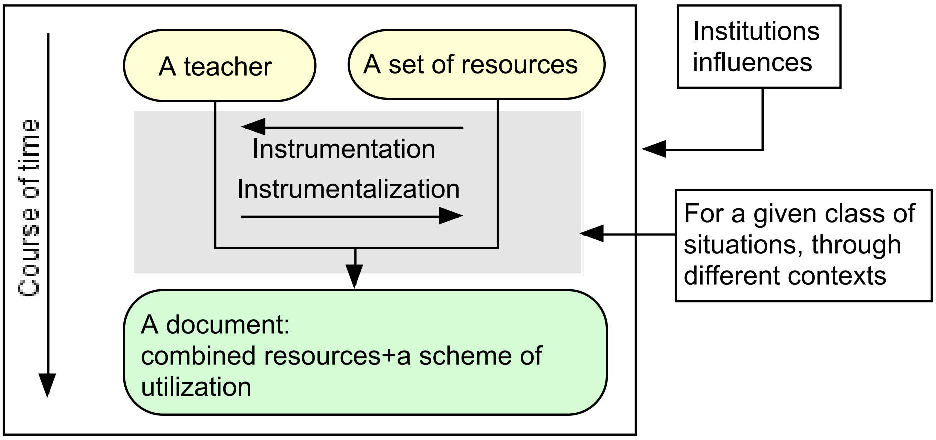

Figure 1. Schematic representation of a documentational genesis.

DOCUMENTATIONAL GENESES: TWO ILLUSTRATIVE EXAMPLES

We use a first case study (figure 2) coming from our interviews to illustrate the distinction between a set of resources and a document, and precise in particular which kinds of operational invariants can intervene in the teachers' professional schemes.

Marie-Pierre is aged 40. She is teaching at secondary school for 14 years, from grade 6 to 9. Marie-Pierre is involved in collective work within an IREM (Institute for Research on Mathematics Teaching) group; she does not have institutional responsibilities; she has a strong degree of ICT integration. Marie-Pierre uses dynamic geometry systems, spreadsheets, and many online resources (e-exercises and mathematics history websites in particular). She has a digital version of the class textbook. Marie-Pierre has an interactive whiteboard in her classroom for three years, and uses it in each of her courses. For the introduction of the circle's area in grade 7, she starts in class by using a website comprising historical references (Archimedes using circular sections to link the perimeter and the area of a circle) and displaying an animation of the circle unfolding and transforming into a triangle (roughly, but that point is not discussed). Then she presents her own course, based on an extract of the class digital textbook. She complements as usual the files displayed on the whiteboard by writing additional comments and explanations, highlighting important expressions etc.

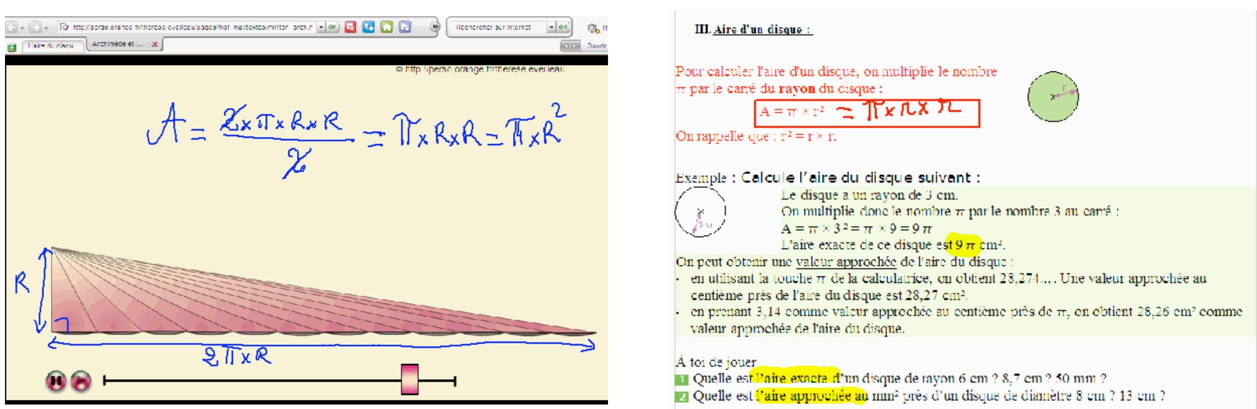

Figure 2. Marie-Pierre, example of a lesson introducing the circle's area

For the class of situations: "preparing a lesson about the circle's area in grade 7" (figure 2), Marie-Pierre draws on a set of resources comprising the interactive whiteboard, a website[12], a digital textbook, and a hard copy of it. The official curriculum texts, about the circle area, only state that "an inquiry-based approach permits to check the area formula", with no more details. The digital textbook proposes an introductory activity with a digital geometry software: drawing circles, and displaying their areas. Several radius are tested, the radius' square and the corresponding area are noted by the students in a table, and they are asked to observe that they obtain an(approximate) ratio table. But Marie-Pierre prefers to draw on the website animated picture (both choices correspond more to an observation activity for the students than to an inquiry-based approach, but we will not discuss this aspect here). So, we claim that she has developed a scheme of utilization of this set of resources, structured by several operational invariants. These invariants are professional beliefs that we infer from our data:

-"A new area formula must be justified by an animation showing a cutting and recombining of the pieces to form a figure whose area is known". This operational invariant concerns all the areas introduced, it also intervenes in the document corresponding to the introduction of the triangle's area for example.

-"The circle's area must be linked with a previously known area: the triangle"; "The circle's area must be linked with the circle's perimeter". These operational invariants are related with the precise mathematical content of the lesson, they were built along the years, with different grade 7 classes (Marie-Pierre uses this website's animation for three years, with two grade 7 classes each year).

We do not assert that these operational invariants were not present among Marie-Pierre's professional knowledge before her integration of the interactive whiteboard. But the possibility to display an animation on a website, to complement it by writing additional explanations, to go back to a previous state of the board to link the "official" formula with what has been observed, yielded a document integrating these operational invariants. And we claim that the development of this document is likely to reinforce, in particular, the above presented beliefs. The operational invariants are both driving forces and outcomes of the teacher's activity.

We use a second case study (figure 3) to emphasize an important aspect of the documentational geneses: documentational genesis must not be considered as a transformation with a set of resources as input, and a document as output. It is an ongoing process. Rabardel & Bourmaud (2005) claim that the design *continues in usage*. We consider here accordingly that a document developed from a set of resources provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, we speak of a *dialectical* relationship between resources and documents.

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[12] http://pagesperso-orange.fr/therese.eveilleau/pages/hist_mat/textes/mirliton.htm

Marie-Françoise is aged 55; she is involved in collective work within an IREM group; she has institutional responsibilities as in-service teacher trainer; and a strong degree of ICT integration. She works with students from grade 10 to 12. She organizes for them 'research narratives': problem solving sessions, where students work in groups on a problem and write down their own 'research narratives' (both solutions and research processes). Thus one class of situations, for Marie-Françoise is 'elaborating open problems for research narratives sessions'. For this class of situations, she draws on a set of resources comprising various websites, but also personal existing resources, colleagues' ideas, etc.; but as she told us: "There is the problem and the way you enact it, because students are free to invent things, and afterwards we benefit from the richness of all these ideas, and you can build on it."; it appears thus that the research narrative session depends on the students' ideas and propositions, thus the design goes on in class. Moreover, the class sessions provide new resources: the students' research narrative, that Marie-Françoise collects, and saves in a new binder, aiming to enrich the next document built on the same open problem.

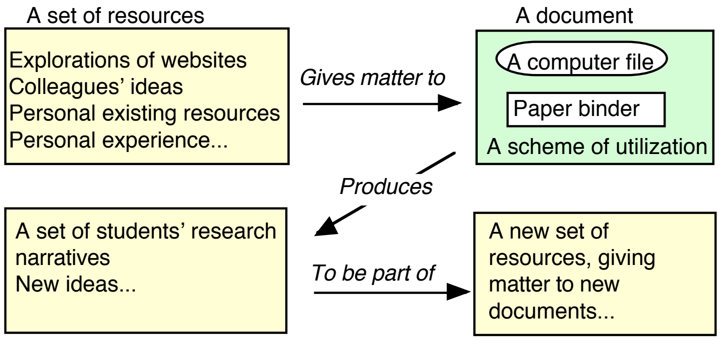


Figure 3. An illustration of the resources/document dialectical relationship.

The resources evolve, are modified, combined; documents develop along geneses and bear new resources (figure 3) etc. We consider that these processes are part of teachers' professional evolutions, and play a crucial role in them.

DOCUMENTATION SYSTEMS AND PROFESSIONAL DEVELOPMENT

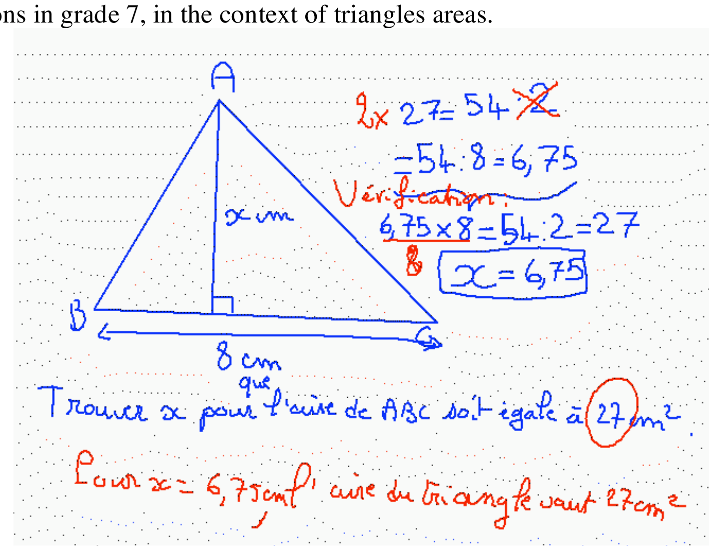
According to Rabardel (2005), professional activity has a double dimension. Obviously a productive dimension: the outcome of the work done. But the activity also entails a modification of the subject's professional practice and beliefs, within a constructive dimension. Naturally, this modification influences further production processes: the productive/constructive relationship has a dialectical nature.

Teachers' documentation work is the driving force behind documentational geneses, thus it yields productive and constructive professional changes. Literature about teachers' professional change raises the question of the articulation between change of practice and change of knowledge and beliefs. We consider that both are strongly intertwined (e.g., Cooney, 2001). The documentational geneses provide a specific view of this relationship. Working with resources, for the same class of situations across different contexts, leads to the development of a scheme, and in particular of rules of action (professional practice features) and of operational invariants (professional implicit knowledge or beliefs). And naturally these schemes influence the subsequent documentation work. All kinds of professional knowledge are concerned by these processes, the evolutions they generate are not curtailed to curricular knowledge (Schulman, 1986). Thus, studying teachers' documents can be considered as a specific way to study teachers' professional development.

According to Rabardel and Bourmaud (2005), the instruments developed by a subject in his/her professional activity constitute a system, whose structure corresponds to the structure of the subject's professional activity. We hypothesize here similarly that a given teacher develops a structured documentation system.

Let us go back to the example of Marie-Pierre evoked above.

Marie-Pierre keeps all her "paperboards" (digital files with images corresponding to the successive states of the board). She uses these paperboards at the beginning of a new session, to recall what has been written, by herself or by her students, during the preceding session. On her laptop, Marie-Pierre has one folder for each class level. Each of these folders contains one file with the whole year's schedule, and lessons folders for each mathematical theme. The paperboards are inside the lessons folders. The interactive whiteboard screen below corresponds to the introduction of equations in grade 7, in the context of triangles areas.



(*Translation:* Find x such that ABC area equals 27 cm^2. or x = 6.75cm, the triangle's area is 27 cm^2).

Figure 4. A view on Marie-Pierre's documents.

Marie-Pierre's files organization on her computer (figure 4), and her statements during the interviews, clearly indicate articulations between her documents. The document whose material component is the year schedule naturally influenced her lesson preparations; but on the opposite, the documents she developed for lessons preparations during previous years certainly intervened in the schedule design. Documents corresponding to connected mathematical themes are also connected. For a given lesson, the students' interventions can contribute to generate operational invariants that will intervene in preparations about other related topics.

A teacher's documents constitute a system, whose organization matches the organization of her professional activity. The evolutions of this documentation system correspond to professional evolutions.

Integration of new materials are, most of the time, visible evolutions of the professional practice, and of the documentation system (in the approach we propose, this integration means that a new material is inserted in a set of resources involved in the development of a document). When Marie-Pierre integrates the interactive whiteboard in her courses, it entails a productive dimension: she now teaches with this whiteboard. But it also yields other changes of her practice: now she makes more links with previous sessions, in particular recalling students productions is now present in her orchestration choices. And it even generates changes in her professional beliefs, for example about the possible participation of students to her teaching. She seems to have developed a operational invariant like: "a good way to launch a lesson is to recall students' interventions done during the preceding lesson".

The integration of new material is always connected with professional practice and professional beliefs evolutions. But professional evolutions do not always correspond to integration of new material, and the same is true for documentation systems evolutions. For example, Arnaud (47 years old, no collective involvement, institutional responsibilities as in-service teacher trainer, low degree of ICT integration) presented during his interview "help sheets" that he designed years ago for students encountering specific difficulties. He now uses the same sheets as exercises for the whole class; thus while no changes can be observed in the material, the action rules associated evolved.

Integration of new material remains an important issue, especially when the focus is on ICT. The study of a given teacher's documentation system also provides insights in the reasons for the integration or non-integration of a given material. The integration depends indeed on the possibility for this material to be involved in the development of a document that will articulate with others within the documentation system. For many years Marie-Pierre prepares her courses as digital files, she uses dynamic geometry software, and online resources; the interactive whiteboard articulates with this material. Moreover, Marie-Pierre is convinced of the necessity of fostering students' interventions, and even of including these in the written courses, and the interactive whiteboard matches this conviction. Possible material articulations are important; but other types of articulations must be taken into account, and the integration of new material also strongly depends on operational invariants, thus on teachers' professional knowledge and beliefs.

CONCLUSION

This paper is related with the second theme of WG7: *Interaction between resources and teachers' professional practice*. It introduces a conceptualization of teachers' interactions with resources and of the associated professional development. Here we just presented the first concepts of a theory whose elaboration is still in progress. Studying teachers' documentation work requires to set specific methodologies, permitting to capture their work in and out of class, to precise their professional beliefs, and to follow long-term processes: it is the main goal of our research. We did not discuss here the very important issue of collective documentation work, which causes particular processes. Its study raises the question of collective documentational genesis and documentation systems, and raises new theoretical needs. The documentational approach we propose also needs to be confronted with other teaching contexts: primary school, tertiary level; diverse countries; and also outside the field of mathematics. Further research is clearly needed; the present evolutions of digital resources make it a major challenge for the studies of teachers' professional evolutions.

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