THE TEACHER'S USE OF ICT TOOLS IN THE CLASSROOM AFTER A SEMIOTIC MEDIATION APPROACH

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The issue of the teacher's role in exploiting the potentialities of ICT tools in classroom is more and more raising the interest of our community. We approach this issue from the Semiotic Mediation perspective, which assigns a crucial importance to the teacher in using ICT tools in the classroom. In the report we describe a Teaching Sequence centred on the use of the tool Casyopée and inspired by the Theory of Semiotic Mediation. Then we focus on the teachers' use of the tool with respect to the orchestration of collective activities and present an on-going analysis of her actions.

INTRODUCTION

Recent research points out a wide-spread sense of dissatisfaction with the degree of integration of technological tools in mathematic classrooms. Kynigos et al. observe that so far one did not succeed to exploit the ICT potential suggested by research in the 80s and the 90s and denounce that "the changes promised by the case study experiences have not really been noticed beyond the empirical evidence given by the studies themselves" (Kynigos et al. 2007, p.1332).

The acknowledgement of the existing gap between the research results on the use of technology in the mathematical learning and the little use of these technologies in the real classroom led recently to reconsider the importance of the teacher in a technology-rich learning environment, and to investigate ways of supporting teachers to use technological tools.

Those "teacher-centred" studies have been developed from different perspective and address different aspects, for instance: teacher education (Wilson, 2005), teachers' ideals and aspirations regarding the use of ICT (Ruthven, 2007), teacher's role in exploiting the potentialities of ICT tools in the classroom.

With that respect, as Trouche underlines, most studies refer to the importance of teachers' guide or assistance to students' activities with the technology (Trouche, 2005). Trouche himself emphasizes the need of taking into account the teacher's actions with ICT. For that purpose he introduces the notion of "instrumental orchestration", that is the intentional systematic organization of both artefacts and humans (students, teachers…) of a learning environment for guiding the instrumental geneses for students (ibidem, p.126).

Within this approach the teacher is taken into account insofar as a guide for the constitution of mathematical instruments.

As we will argue in the next section, guiding the constitution of mathematical instruments does not exhaust the teacher's possible use of ICT. In fact ICT tools can be used by the teacher (a) for developing shared meanings having an explicit formulation, de-contextualized with respect to the ICT tool itself and its actual, recognizable and acceptable in respect to mathematicians' community, and (b) for fostering students' consciousness-raising of those meanings. The Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008) takes charge of that dimension.

In this report, we present an analysis of the teacher's use of an ICT tool within the frame of the Theory of Semiotic Mediation. More precisely we focus on the teacher's promotion and management of collective discussions. But a systematic discussion of the role of the teacher or a classification of her possible actions is out of the goals of the present paper. The context is a teaching sequence, inspired by the Theory of Semiotic Mediation, and centred on the use of the tool Casyopée. Both the teaching sequence and the tool are presented in the next sections, after recalling some basic assumptions of the Theory of Semiotic Mediation.

THE THEORY OF SEMIOTIC MEDIATION

Assuming a Vygotskijan perspective Bartolini Bussi and Mariotti put into evidence that the use of an artefact for accomplishing a (mathematical) task in a social context may lead to the production of signs, which, on the one hand, are related to the actual use of the artefact (the so called **artefact-signs**), and, on the other one, may be related to the (mathematical) knowledge relevant to the use of the artefact and to the task. As obvious, this knowledge is expressed through a shared system of signs, the mathematical signs. The complex of relationships among use of the artefact, accomplishment of the task, artefact-signs and mathematical signs, is called the **semiotic potential** of the artefact with respect to the given task.

Hence, in a mathematics class context, when using an artefact for accomplishing a task, students can be led to produce signs which can be put in relationship with mathematical signs. But, as the authors clearly state, the construction of such relationship is not a spontaneous process for students. On the contrary it should be assumed as an explicit educational aim by the teacher. In fact the teacher can intentionally orient her/his own action towards the promotion of the evolution of signs expressing the relationship between the artefact and tasks into signs expressing the relationship between the artefact and knowledge.

According to the Theory of Semiotic Mediation, the evolution of students' personal signs towards the desired mathematical signs is fostered by iteration of **didactic cycles** (Fig.1) encompassing the following semiotic activities:

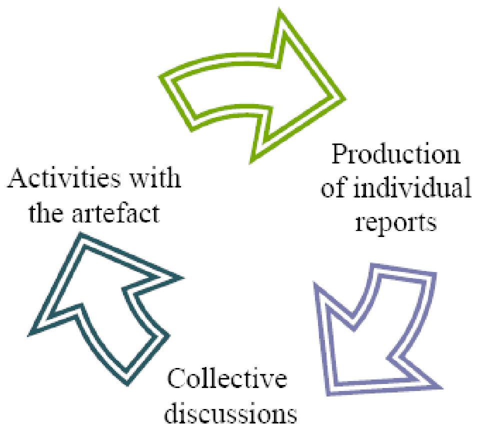

Fig. 1. Didactical Cycle

- activities with the artefact for accomplishing given tasks: students work in pair or small groups and are asked to produce common solutions. That entails the production of shared signs;
- students' individual production of reports on the class activity which entails personal and delayed rethinking about the activity with the artefact and individual production of signs;
- classroom collective discussion orchestrated by the teacher

The action of the teacher is crucial at each step of the didactic cycle. In fact the teacher has to design tasks which could favour the unfolding of the semiotic potential of the artefact, observe students' activity with the artefact, collect and analyse students' written solutions and home reports in particular posing attention to the signs which emerge in the solution, then, basing on her analysis of students written productions, she has to design and manage the classroom discussion in a way to foster the evolution towards the desired mathematical signs.

The Theory of Semiotic Mediation offers not only a frame for designing teaching interventions based on the use of ICT, but also a lens through which semiotic processes, which take place in the classroom, can be analysed (for a more exhaustive view, see Bartolini Bussi and Mariotti, 2008).

CASYOPÉE

Casyopée (Lagrange and Gelis 2008) is constituted by two main environments which can "communicate" and "interact" between them: an Algebraic Environment and a Dynamic Geometry Environment (though the designers' objective was not to develop a complete CAS or a complete DGE). Possible interactions between the two environments are supported through a third environment, the so called "Geometric Calculation". Without entering the details of Casyopée functioning, we can illustrate it through the following example.

If one has two variable geometrical objects in the DGE linked through a functional relationship (e.g. the side of a square and the square itself), Casyopée supports the user in associating algebraic variables to the geometrical variables and building an algebraic expression for the function (e.g. the function linking the measure of the length of the side, as independent variable, and the measure of the area of the square, as dependent variable). The generated algebraic variables and functions can be exported in the Algebraic Environment, and then explored and manipulated.

DESCRIPTION OF THE TEACHING EXPERIMENT

The Theory of Semiotic Mediation shaped both the design and the analysis of the teaching experiment carried out. In this chapter, we briefly describe the design.

Educational Goals of the designed teaching sequence.

The design of the teaching intervention started from the analysis of the semiotic potential of the tools of Casyopée. That analysis led us to identify two main educational goals: fostering the evolution of students' personal signs towards

1. the mathematical signs of function as co-variation and thus consolidate (or enrich) the meanings of function they have already appropriated, that entails also the notions of variable, domain of a variables…;
2. the mathematical meanings related to the processes characterizing the algebraic modelling of geometrical situation.

Description of the teaching sequence

According to our planning the whole teaching sequence is composed of 7 sessions which could be realized over 11 school hours.

The whole teaching sequence is structured in didactical cycles: activities with Casyopée alternate with class discussions, and at the end of each session students are required to produce reports on the class activity for homework.

The familiarization session is designed as a set of tasks and aims at providing students with an overview of Cayopée features and guiding students to observe and reflect upon the "effects" of their interaction with the tool itself, e.g.:

Could you choose a variable acceptable by Casyopée and click on the "validate" button? Describe how the window "Geometric Calculation" change did after clicking on the button. Which new button appeared?

Besides familiarization, the designed activities with Casyopée consist of coping with "complex" optimization problems formulated in a geometrical setting and posed in generic terms, e.g.:

Given a triangle, what is the maximum value of the area of a rectangle inscribed in the triangle? Find a rectangle whose area has the maximum possible value.

The aim is to elaborate on those problems so to reveal and unravel the complexity and put into evidence step by step the specific mathematical meanings at stake.

The diagram (Fig. 2) depicts the structure of the teaching sequence: the cyclic nature of the process, which develops in spirals, is rendered through the boxing of the cycles themselves.

Implementation and data collection

With some differences, the teaching sequence was implemented in 4 different classes (3 different teachers): two 13 grade classes and a 12 grade class of two Scientific High Schools, and a 13 grade class of Technical School with Scientific Curriculum.

Different kinds of data were collected: students' written productions; screen, audio and video recordings, and Casyopée log files. The analysis presented below is based on the verbatim transcription of the video recordings of the classroom discussions.

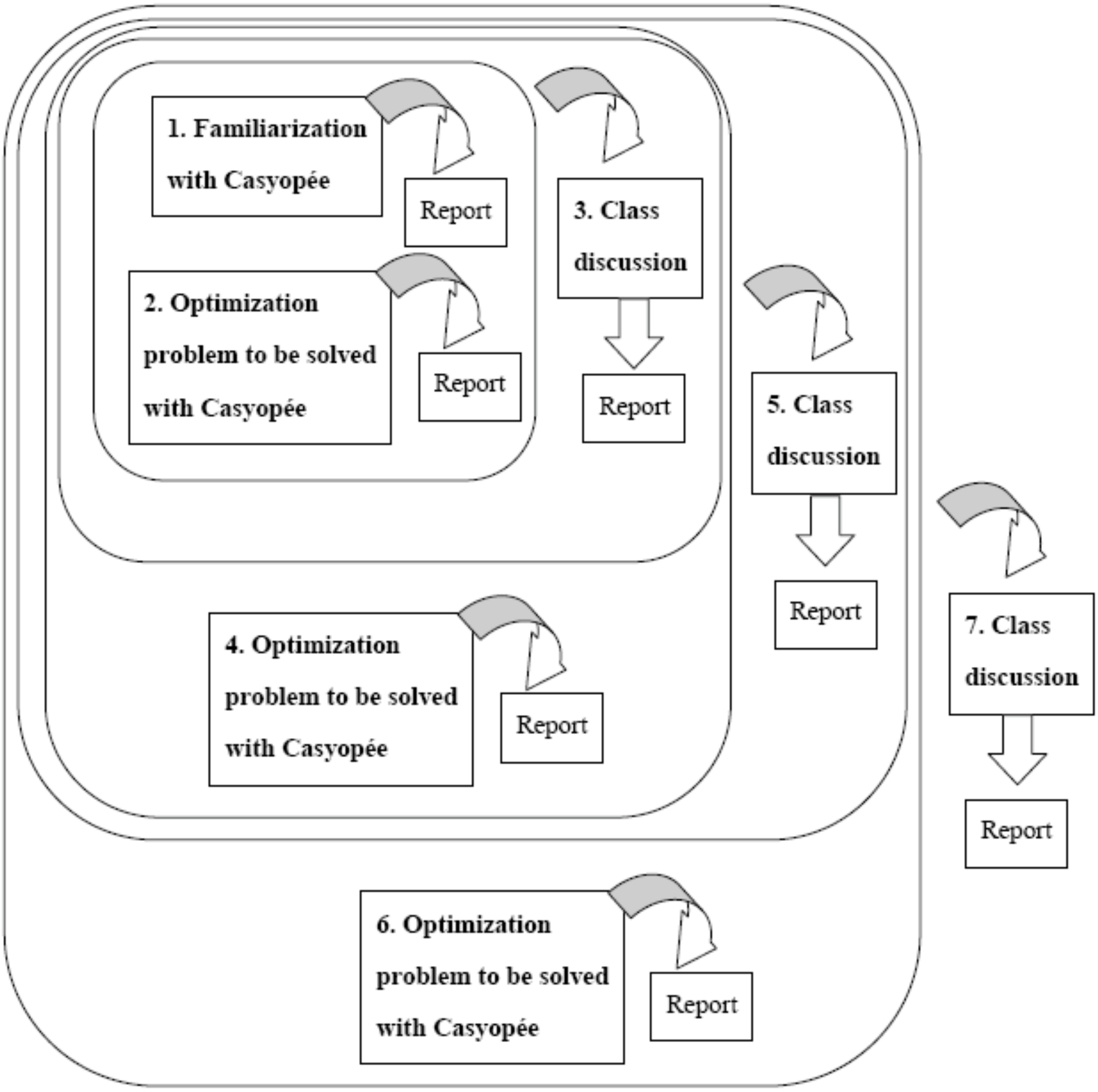

Fig. 2. Outline of the Teaching

ANALYSIS OF THE TEACHER'S ACTIONS

According to the theory of Semiotic Mediation, the teacher's action should aim at promoting the evolution of students' personal signs towards mathematical signs. Such evolution can be described in terms of *semiotic chains,* or chains of signification to use Walkerdine's terminology, that is:

"particular chain of relations of signification, in which the external reference is suppressed and yet held there by its place in a gradually shifting signifying chain." (Walkerdine, 1990, p.121).

The following excerpt is drawn from the transcript of the class discussion held in the 5th session. It shows an example of how artefacts signs are produced in relation to the use of the artefact, and how they may evolve during the discussion. We first go quickly through the excerpt showing the evolution of signs, then we will analyse how the teacher contributes to this evolution.

1. Teacher A: "Which are the main points to approach this kind of problem? Which kind of problem did we deal with? […] What is an important thing you should do now? To see the general aspects and apply them for solving possible more problems with or without the software, […] the software guided you proposing specific points to focus on.[…]"

2. Cor: "[…] First of all we had to choose the triangle by giving coordinates"

[Students recall the steps to represent the geometrical situation within Casyopée DGE]

5. Luc: "But you have to choose a mobile point, first […]"

6. Teacher A: "Does everybody agree?[…]How would you label this first part? […]"

7. Students: "Setting up"

8. Teacher A: "Luc has just highlighted something […] do you see anything similar between the two problems?"

9. Sam: "One has always to take a free point which varies, in this case, the areas considered […]"

10. Teacher A: "Then we have a figure which is…"

11. Students: "Mobile."

12. Teacher A: "Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? […]"

13. And: "The observation of the figure would let us see… we need to study that figure and observe what the shift of the variable causes…"

14. Teacher A: "Ok, then? Everybody did that, isn't it?"

15. Sil: "We computed the area of the triangle and of the parallelogram, we summed them, and by shifting the mobile point one observed as [the sum of the areas] varied […]"

Focusing on students' signs, one can notice:

- Elements of a collectively constructed *semiotic chain*, in which a connection is established between artefact signs ("mobile point") and mathematical signs ("variable"). The elements of this semiotic chain are: "movable point" (item 5), "free point" (item 9), "variable" (item 13), and "movable point" (item 15). It is worth noticing the two directions: from the artefact sign ("mobile point") to the mathematical sign ("variable") and vice versa. That semiotic chain shows: (a) students' recognition that geometrical objects can be considered (can be treated, can act) as variables, and (b) the enrichment of students' meanings of variable to include meanings related to "movement".

- Elements of a collectively constructed *semiotic chain,* in which the meaning of function as a relation of co-variation of two variables emerges. The elements of this semiotic chain are: "a free point which varies […] the areas" (item 9), "the shift of the variable causes" (item 13), "by shifting the movable point, one observed as [the sum of the areas] varied" (item 15).

Analysis of the Teachers' orchestration of the discussion.

We reconsider the excerpt previously analysed form the point of view of the signs produced and used by students. Here we focus on how the teacher's actions fuel the discussion, foster the production of artefacts signs in relation to the use of the artefact, and create the conditions for their evolution during the discussion.

1. Teacher A: "Which are <u>the main points to approach</u> this kind of problem? Which <u>kind of problem</u> did we deal with? […] What is an important thing you should do now? <u>To see the general aspects</u> and apply them for solving possible more problems with or <u>without the software</u>, […] <u>the software guided you proposing specific points to focus on</u>.[…]"

The teacher starts the discussion by making explicit its objectives: to arrive at a shared and de-contextualized formulation of the different mathematical notions at stake ("to see the general aspects and apply them for solving possible more problems with or without the software").

In order to do that, the teacher asks students to recall the problem dealt with in the previous section and to report on the solutions they produced. She explicitly orients the discussion towards the specification of the main phases of the solution of the problem, asking students to look for similarities between the two problems addressed so far and between the strategies enacted to solve them.

While asking students to do that, the teacher suggests to refer to (or to remind) the use of the DDA. The suggestion to explicitly refer to the use of Casyopée facilitates the production and use of artefact-signs and the unfolding of the semiotic potential.

5. Luc: "But you have to <u>choose a mobile point</u>, first […]"

…

8. Teacher A: "<u>Luc has just highlighted something</u> […] do <u>you see anything similar between the two problems</u>?"
9. Sam: "One has always to take a <u>free point</u> <u>which varies</u>, in this case, <u>the areas</u> considered […]"

Following the teacher's request, students collectively report on their work with Casyopée. That leads to the production of the artefact sign "mobile point" (out of the others) (item 5). The sign "mobile point" is clearly related to the task and the use of Casyopée for accomplishing it. At the same time it may be related to the mathematical knowledge at stake: the notion of variable. There are several possibilities for the subsequent development of the discussion: one could orient the discussion towards the distinction between mobile and variable, towards the specification of other variable elements, discussion towards the distinction between algebraic or numerical variable and geometrical variable, towards the recognition of the aspects of co-variation between the variable elements of the geometrical figure, towards the distinction between independent and dependent variable.

Certainly, the teacher's intervention is needed both to drive the attention of the class towards the sign introduced by Luc and to orient the discussion. The teacher is aware of that and intentionally emphasizes Luc's contribution to the discussion (item 8). At one time, she requires to generalize so to foster a de-contextualization from the specific problems faced and strategies enacted, and to provide the possibilities for the evolution of personal signs to initiate.

After the teacher's intervention, Sam (item 9) echoes Luc's words. But she uses the sign "free point" instead of "mobile point", and introduces the consideration of other variable elements ("areas") also emphasizing the existence of a link between them ("free point which varies […] the areas"). Those are the first elements of the two semiotic chains described in the previous section.

10. Teacher A: "Then we have a <u>figure</u> which is…"
11. Students: "<u>Mobile</u>."
12. Teacher A: "<u>Mobile, dynamical</u>. Let us pass to the second phase. Andrea, which is the next phase? […]"
13. And: "The <u>observation of the figure</u> would let us see… we need to study that <u>figure and observe what the shift of the variable causes</u>…"

Sam's contribution (item 9) ends with the reference to variable areas. That could prematurely move the discussion towards the consideration of algebraic or numerical aspects, without giving time to elaborate on variable and variation in the geometric setting. In order to contrast this risk, the teacher introduces the term "figure" (item 10) which has the effect of keeping students' attention still on the geometrical objects. In addition the teacher fuels the discussion echoing students and, thus, emphasizing the reference to the dynamical aspects (item 12), which nurtures the construction of the semiotic chains on variation and co-variation.

And, whose intervention is stimulated by the teacher, echoes the use of the sign "figure" and makes explicit exactly the co-variation between the geometrical objects in focus. She also introduces the sign "variable" so establishing a connection between the artefact sign "mobile point" and the sign "variable".

We are not claiming that the evolution towards the target mathematical signs is completed: a shared and de-contextualized formulation of the different mathematical notions at stake is not reached yet, as witnessed by Sil's words (item 15), who still makes reference to the use of the artefact in her speech.

14. Teacher A: "Ok, then? Everybody did that, isn't it?"

15. Sil: "We computed the area of the triangle and of the parallelogram, we summed them, and by shifting the mobile point one observed as [the sum of the areas] varied […]"

The above analysis puts into evidence a number of interventions of the teachers who succeeds in exploiting the semiotic potential of Casyopée, and thus in making the class progress towards the achievement of the designed educational goals.

One can find also episodes in which the teacher' action is not so efficient. The following excerpt is drawn from a discussion held in another class and orchestrated by a different teacher, and it shows an episode in which the teacher does not succeed to exploit the potentialities of the students' interventions. Chi countered the sign "variable" with the sign "variable point" so offering the possibility to dwell on the relationship between not measurable geometrical variables and measurable geometrical variables. The specification of this distinction was considered a key aspect of algebraic modeling, and as such highly pertinent to the designed educational goals. The teacher does not seize the occasion and does not take any action to fuel the discussion on that, she was probably aiming at orienting the discussion along a different direction.

184. Chi: "we put CD as variable, and not by chance CD, in fact we used a fixed point, C, and a variable point on the segment, D"

185. Teacher B: "well, the underpinning idea is to link numbers, and, […] having observed a link between the position of the point D and […] the area of the rectangle […] a link is established between a geometrical world and an algebraic world"

That witnesses the difficulty of mobilizing strategies to foster the evolution of students' signs. One has to constantly keep the finger on the pulse of the discussion and of its possible development. In fact the evolution of students' signs depends on extemporary stimuli asking for a number of decisions on the spot.

CONCLUSIONS

The analysis carried out in the paper confirms the crucial role of the teacher in technology-rich learning environments. In particular, such role may (and should from our perspective) go beyond that of assistant or guide for students' instrumental genesis process. In fact through her interventions the teacher promotes and guides the development of the class discussion, so to foster the production and the evolution of students' signs towards the target mathematical signs, and to facilitate students' consciousness-raising of the mathematical meanings at stake.

Certainly we are aware that the analysis presented is still at a phenomenological level. There is an emerging need for elaborating a more specific model for analysing the teacher' semiotic actions. But there is not only the need of developing tools for finer analysis. We showed an episode witnessing the difficulty of mobilizing strategies to foster the evolution of students' signs. Currently, the Theory of Semiotic Mediation does not equally support analysis and planning. Due to the richness of a class discussion and the number of extemporary stimuli which could emerge, one cannot foresee the exact development of the discussion. That makes the teacher's role still more crucial. Nevertheless there is the need of an effort for elaborating more specific theoretical tools for supporting the a-priori design of classroom discussion. All this is also relevant to the more generic issue of teacher's formation.

NOTES

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NAVIGATION IN GEOGRAPHICAL SPACE

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This study is part of the ReMath project (Remath' – Representing Mathematics with Digital Media FP6, IST-4, STREP 026751 (2005 – 2008), http://remath.cti.gr. Twenty four 10th Grade students participated in a constructivist teaching experiment, the aim of which was to investigate children's constructions of mathematical meanings concerning the concept of function while navigating within 3d large scale spaces. The results showed that the utilization of the new representations provided by the dynamic digital media such as Cruislet could reform the way that mathematical concepts are presented in the curricula and possibly approach these mathematical notions through meaningful situations. The new representations provide the opportunity to introduce and study mathematical notions not as isolated entities but rather as interconnected functionalities of meaningful real – life situations.

Functions are a central feature of mathematics curricula, both past and present. Many research studies indicate students' difficulty in understanding the concept of functions. This difficulty comes from a) the static media used to represent the concept, b) the introduction of function mainly as a mapping between sets in conventional curricula, c) the use of formalisation and function graphs as the only representations. With digital media, students can dynamically manipulate informal representations of function defined as co-variation and rate of change, which is an interesting and powerful mathematical concept. Tall(1996) points out a fundamental fault-line in "calculus" courses which attempt to build on formal definitions and theorems from the beginning. Moreover, he suggests that enactive sensations of moving objects may give a sense that "continuous" change implies the existence of a "rate of change", in the sense of relating the theoretically different formal definitions of continuity and differentiability. The enactive experiences provide an intuitive basis for elementary calculus built with numeric, symbolic and visual representations.

The 'Cruislet' environment is a state-of-the-art dynamic digital artefact that has been designed and developed within the Eu ReMath project. It is designed for mathematically driven navigations in virtual 3d geographical spaces and is comprised of two interdependent representational systems for defining a displacement in 3d space, a spherical coordinate and a geographical coordinate system. We consider that the new representations enabled by digital media such as Cruislet can place mathematical concepts in a central role for both controlling and measuring the behaviours of objects and entities in virtual 3d environments. The notion of *navigational mathematics* is used to describe the mathematical concepts that are embedded and the mathematical abilities the development of which is supported within the Cruislet microworld. In this study we focus on how students using spherical and geographical systems of reference in Cruislet construct meanings about the concept of function.

THEORETICAL FRAMEWORK

A number of research studies suggest that students of all grades, even undergraduate students, have difficulties modelling functional relationships of situations involving the rate of change of one variable as it continuously varies in a dependent relationship with another variable (Carlson et all, 2002; Carlson, 1998, Monk & Nemirovsky, 1994). This ability is essential for interpreting models of dynamic events and foundational for understanding major concepts of calculus and differential equations. On the other hand, the VisualMath curriculum (Yerushalmy & Shternberg, 2001) is an a example of a function based curriculum that involves the moving across multiple views of symbols, graphs, and functions. VisualMath uses specially designed software environments such as simulations' software, or other modelling tools that include dynamic forms of representations of computational processes. Yerushalmy (2004) suggests that such emphasis on modeling offers students means and tools to reason about differences and variations (rate of change). Moreover, .Kaput and Roschelle (1998) using computer simulations study aspects of calculus at an earlier stage. These simulations (MBL tools), permit the study of change and the ways it relates to the qualities of the situation. In their study Nemirovsky, Kaput and Roschelle (1998) show that young children can use the rate of change as a way to explore functional understanding. In studying the process of the understanding of dynamic functional relationships, Thompson (1994) has suggested that the concept of rate is foundational.

Confrey and Smith (1994) choose the concept of rate of change as an entry to thinking about functions. They introduce introduce two general approaches to creating and conceptualizing functional relationships, a correspondence and a covariation approach. They suggest that “a covariational approach to functions makes the rate of change concept more visible and at the same time, more critical (p. 138). They explicate a notion of covariation that entails moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable.

Moreover, Carlson, Larsen and Jacobs (2001) stress the importance of covariational reasoning as an important ability for interpreting, describing and representing the behavior of dynamic function event. They consider covariational reasoning to be the cognitive ability involved in coordinating images of two varying quantities and attending to the ways in which they change in relation to each other. On the same line, Saldanha and Thompson (1998) introduced a theory of developmental images of covariation. In particular, they considered possible imagistic foundations for someone’s ability to see covariation. Carlson et all (2001) in their study exploring the role of covariational reasoning in the development of the concepts of limit and accumulation, suggest a framework including five categories of mental actions of covariational reasoning:

1. An image of two variables changing simultaneously
2. A loosely coordinated image of how the variables are changing with respect to each other
3. An image of an amount of change of one variable while considering changes in discrete amounts of the other variable
4. An image of the average rate-of-change of the function with uniform increments of change in the input variable
5. An image of the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

The proposed covariation framework contains five distinct developmental levels of mental actions. Using this particular framework we will try to classify students' covariational reasoning while studying navigation within the context of Cruislet microworld. We consider navigation as a dynamic function event. The function's independent variable is the geographical coordinates of the position of the first aeroplane, which students are asked to navigate, while the dependent variable is the geographical coordinates of the position of the second aeroplane.

Our approach to learning promotes investigation through the design of activities that offer a research framework to investigate purposeful ways that allow children to appreciate the utility of mathematical ideas (Ainley & Pratt, 2002). In this context, our approach is to design tasks for either exclusively mathematical activities or multi-domain projects containing a mathematical element within the theme which can be considered as marginalized or obscure within the official mathematics curriculum (Kynigos & Yiannoutsou, 2002, Yiannoutsou & Kynigos, 2004).

TASKS

In the tasks that are included in this teaching experiment, students actually engage with the study of the existence of a rate of change of the displacements of the airplanes which are defined in the geographical coordinate system. In particular the dispacements of two airplanes are relative according to a linear function. This function will be hidden and the students will have to guess it in the first phase of the activity based on repeated moves of aeroplane A and observations of the relative positions and moves of planes A and B. The second phase, the students will be able to change the function of relative motion and play games with objectives they may define for themselves such as move plane A from Athens to Thessaloniki and plane B from Athens to Rhodes and then to Thessaloniki in the same time period.

This scenario is based on the idea of half – baked games, an idea taken from microworld design (Kynigos, 2007). These are games that incorporate an interesting game idea, but they are incomplete *by design* in order to encourage students to change their rules. Students play *and* change them and thus adopt the roles of both player and designer of the game (Kafai, 2006).

Initially, students are asked to study the relation between the two aeroplanes, the rate of change of their displacements and consequently find the linear function (decode the rule of the game). In order to decode "the rule of the game", they should give various values to coordinates (Lat, Long, Height) that define the position of the first plane. They will be encouraged to communicate their observations about the position of the second plane to each other and form conjectures about the relationship between the positions of the two aeroplanes.

In the second phase students are encouraged to build their own rules of the game by changing the function of the relative displacements of the two aeroplanes.

METHODOLOGY

The research methodology is a constructivist teaching experiment along the same lines as described by Cobb, Yackel and Wood (1992).The researcher acts as a teacher interacting with the children aiming to investigate their thinking. The researcher, reflecting on these interactions, tries to interpret children's actions and finally forms models-assumptions concerning their conceptions. These assumptions are evaluated and consequently either verified or revised.

Twenty four (24) students of the 1st grade of upper high school, (aged 15-16 years old) participated in this experiment. Students worked in pairs in the PC lab. Each pair of students worked on the tasks using Cruislet software.

The data consists of audio and screen recordings as well as students' activity sheets and notes. The data was analyzed verbatim in relation to students' interaction with the environment. We have focused particularly on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, in our analysis we use students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.

ANALYSIS

While students were interacting with the Cruislet environment according to the tasks, several meanings emerged regarding the concept of function. We categorise these meanings in clusters that rely upon the concept of function. In particular, there are two major categories:

Domain of numbers

Students navigating an aeroplane in the 3d map of Greece realized that the domain of the geographical coordinates is actually a closed group. The 3d map of Greece is a geographical coordinate system with specific borders. The investigation of the range of the geographical borders as the domain of the function became the subject of study and exploration through the use of the Cruislet functionalities. In particular, students exploited the two different systems of reference and, experimenting with the values of the geographical coordinates, they define the range of the latitude – longitude values. This specific range of values has been considered as the domain of the functions according to which the displacements of the aeroplanes are relative. Although students didn't refer to the values as the domain of the function, we interpret their involvement in finding them, as a mathematical activity regarding the domain of the function.

Students experimented by giving several values to the geographical coordinates of the airplane's position defining at the same time the range of the coordinates' values. In the following episode students are trying to find out the reason for not placing the airplane in a given position.

S1: Why?? It doesn't accept any value. (they gave values in procedure fly1 and the airplane couldn't go).

R: Do you remember what values the lat long coordinates have?

Isn't lat equals 58 isn't correct? (she also speaks to the next team)

S1:It doesn't accept 32 20 100 either.

S2: Greece hasn't got value 20 (student from another team speak ironically to him)

S1: Why? Was the 58 you used correct?

An interesting issue related to the domain of the function, is that the provided representations, i.e. the result of the aeroplane's displacement displayed on the screen, helped students realize that the domain of numbers of the two aeroplanes displaced in relative positions, are strongly dependent. For instance when the first moved to a given position, the second one couldn't go anywhere, but the domain of values was restricted by the first position. In the following episode students realized that the 2nd aeroplane didn't follow them when they flew at a low height. The episode is interesting as it indicates the way students realize the domain of geographical coordinate values that the first aeroplane can take in relation to the other one.

S1: There are some times that it (meaning the other aeroplane) can't follow us.

R: Where? When?

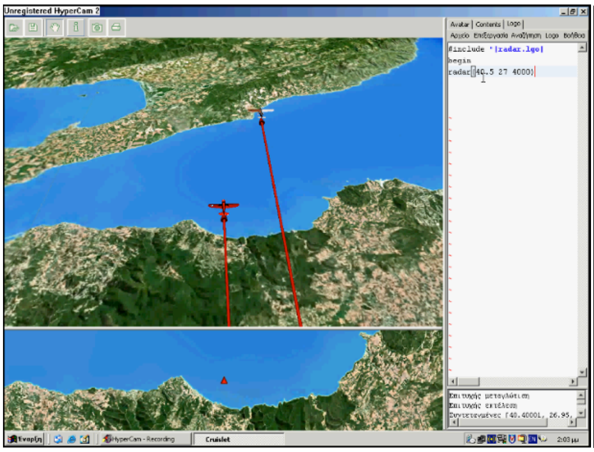
S1: When I'm getting into the sea.

We could say that the characteristics of Cruislet software, such as the visualization of the results of the objects' displacements on the map, acted as a mediator in students' engagement with the domain of function. We have to mention that although the modalities of use of Cruislet software and the communication within the groups didn't reveal that students realized or mentioned anything regarding the concept of function, they did focus on finding ways to move the aeroplanes. In other words, students didn't conceive the values of the coordinates as the domain of the function, although they used it in this way. The interpretation of students' actions relies upon our educational goals, which conceive this as a mathematical activity that was related to the notion of function and particularly, its domain.

Function as covariation

During the implementation of the tasks, students engaged with the notion of function, through their experimentation with the dependent relationship between two aeroplanes' positions, which was defined by a black – box Logo procedure. Trying to find out the hidden function, students' actions and meanings created, suggested they were able to coordinate changes in the direction and the amount of change of the dependent variable in tandem with an imagined change of the independent variable. Our results indicate that students developed covariational reasoning abilities, resulting in viewing the function as covariation.

Initially most of the students expressed the covariation of the aeroplanes' positions using verbal descriptions, such as behind, front, left, etc. as they were visualizing the result of the airplanes' displacements. In the following episode students express the dependent relationship while looking at the result displayed on the screen.



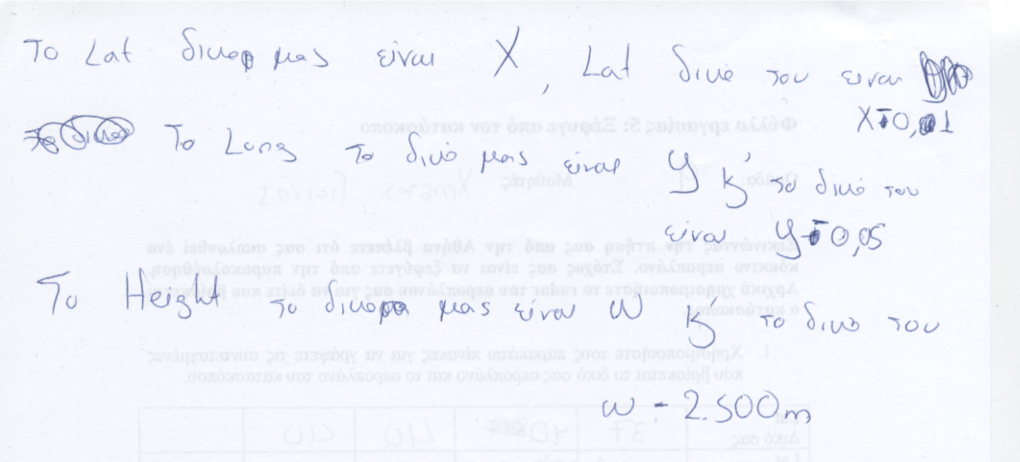
S1: So, he always wants to be close to us on our left.

R: Yes.

S1: And he is beneath, further down to us. Beneath.

S2: And behind.

Students experimented by giving several values to geographical coordinates in Logo and formed conjectures about the correlation between the aeroplanes' positions. Through their interaction with the available representations, they successfully found the dependent relation of the function in each coordinate, resulting in their coming into contact with the concept of function as a local dependency. In fact, one of the teams conceived the relationship among each coordinate as a function, as is obvious in their notes on the activity sheet.



Translation

Our Lat is x, his Lat is x – 0.1
Our Long is y and his is y – 0.05
Our Height is ω and his is ω – 2500m.

It is interesting to mention that students separated latitude and longitude coordinates on the one hand and that of height on the other as they were trying to decode the hidden functional relationship between the airplanes' height coordinates. In particular, they didn't encounter difficulties in decoding latitude and longitude relationship in contrast to their attempts to find the height dependency. Although all three functions regarding coordinates were linear, students conceived the functional relationship between height mainly as proportional, in contrast to latitude and longitude that were comprehended as linear, from the beginning. In the following episode, students endeavor to apply the rate of change of the function to decode the height relationship. As they thought the height coordinates had a proportional relationship, they suggested carrying out a division to find it.

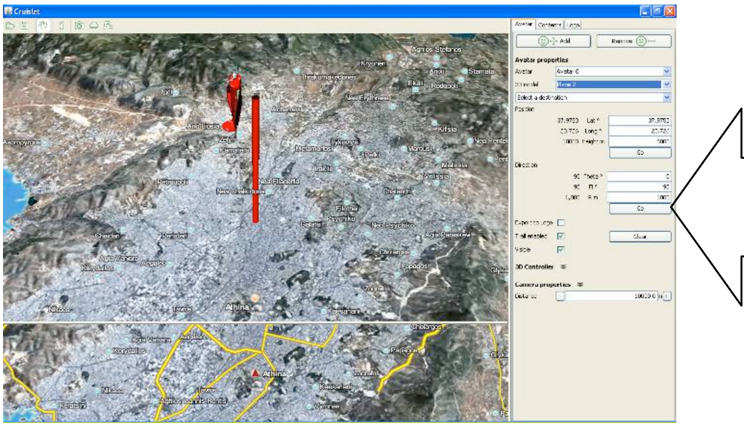
S2: When we go up 1000, he goes up 1000.

R: Do you mean that if we go from 7000 to 8000 he goes from… let's say 2500 to 3500.

S2: He is at… 3000. No. Give me a moment. At 8000 he was at 5500. At 7000 he was at 4500. At 5000 he is as 2500. And then….

S1: We could do the division to see the rate.

An interesting example was the cases of the variation of the height of the aeroplane every time they pushed the button 'go' in spherical coordinates, when they wanted to make a vertical displacement. In particular, by defining the vector of a vertical upward displacement, students observed that height was the only element that changed in the position of the displacement. Through a number of identical displacements students identified and expressed verbally, symbolically and graphically the interdependency between direction functionality and the height of the aeroplane. Students' reasoning: "the more times we push the button GO the higher the aeroplane goes", suggests that students developed a covariational reasoning ability similar to the second level proposed by Carlson et al (2001) of how the variables change with respect to each other. Moreover, the retrospective symbolic type developed by students (h2= h1+1000) indicates that they realized that the rate of change of the height is constant. In the following figures we can see the result displayed on the screen (figure 1) as well as students' writings on the activity sheet (figure 2).



Students' actions:

1. Define the spherical coordinates (theta = 0 fi = 90 R = 1000).
2. Push the "Go" button in "Avatar properties" tab resulting in the vertical displacement of the aeroplane.
3. Watch the displacement of the aeroplane on the GUI.
4. Focus on the changes of the height coordinate.

Figure 1

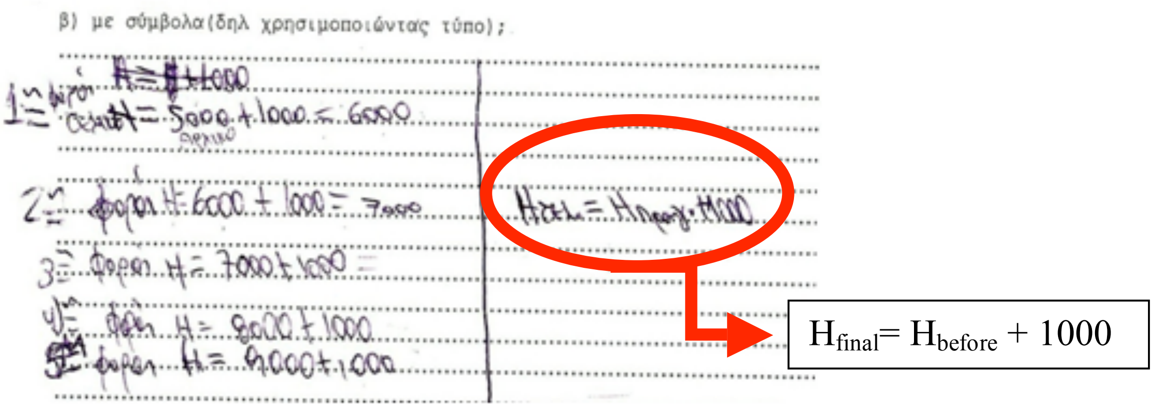


Figure 2

The provided representations of Cruislet software became a vehicle to engage students with concepts related to the concept of function and their expression in a mathematical way. The result of airplanes' displacements on the screen, gave them the chance to realize the dependent relation in 'visual terms' and then express it in mathematical terms. We believe that the results are mainly based on the way that these characteristics were used in the task activity. In particular, the activity was based on the idea of the 'Guess my function' game and the dependent relationship, (built in Logo programming language), was hidden at first. Due to this choice, students focused primarily on the observation of the relative displacements and not on the Logo code underneath it. At the same time perceiving the activity as a game encourages the engagement of students with the activity.

CONCLUSIONS

The study indicated that students exploiting Cruislet functionalities can construct meanings concerning the concept of functions. The provided linked representations (spherical and geographical coordinates), as well as the functionalities of navigating in real 3d large scale spaces actually enable students to explore and build mathematical meanings of the concept of function within a meaningful context. They explore the domain of numbers of a function within a real world situation distanced from the "traditional" formal definitions. On the other hand, they built the concept of function as covariation exploring the variation of the spherical and geographical coordinates. The provided context gave students the opportunity to cope with and explore mathematical concepts at different levels. They navigate within 3d large scale spaces controlling the displacement of an avatar and develop their visualization abilities building mathematical meanings of the concept of function while at the same time they explore the mathematical concepts of spherical and geographical coordinates.

The functionalities of the new digital media such as Cruislet provide a challenging learning context where the different mathematical concepts and mathematical abilities are embedded and interconnected. The role of the teacher becomes crucial in designing mathematical tasks where students' enactive explorations will reveal these mathematical notions and put them under negotiation. In the case of Cruislet, navigational mathematics becomes the core of the mathematical concepts that involves the geographical and spherical coordinate system interconnected with the concept of function and the visualization ability.

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STUDENTS' UTILIZATION SCHEMES OF PANTOGRAPHS FOR GEOMETRICAL TRANSFORMATIONS: A FIRST CLASSIFICATION♦

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Abstract

The activities with the Mathematical Machines are very rich from educational and cognitive points of view. In particular, the use of pantographs has revealed educational potentialities for the acquisition of some important mathematical concepts and for the emergence of argumentation and proving processes, at any school level. In this paper, we propose a cognitive analysis of the processes involved in the manipulation of the mathematical machines, providing a first classification of utilization schemes of pantographs for geometrical transformations. This classification can be efficiently used to observe, describe and analyse cognitive processes involved in the exploration of mathematical properties incorporated in the machines.

Keywords: Mathematical Machines, utilization schemes, pantographs, geometrical transformations and cognitive processes.

Introduction

The Mathematical Machines Laboratory (MMLab: www.mmlab.unimore.it), at the Department of Mathematics in Modena (Italy), is a research centre for the teaching and learning of mathematics by means of instruments (Ayres, 2005; Maschietto, 2005). The name comes from the Mathematical Machines (working reconstruction of many mathematical instruments taken from the history of mathematics), the most important collection of the Laboratory. These machines concern geometry or arithmetic:

> "a geometrical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law"…"an arithmetical machine is a tool that allows the user to perform at least one of the following actions: counting; making calculations; representing numbers" (Bartolini Bussi & Maschietto, 2008).

The MMLab research group carried out various activities with the Mathematical Machines, namely: laboratory sessions in the MMLab, long-term teaching experiments in classrooms, workshops at national and international conferences and also exhibitions (see chapters 2 and 5 of the forthcoming volume by Barbeau and

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Taylor, from ICMI Study n. 16) in collaboration with the members of the association "Macchine Matematiche" (http://associazioni.monet.modena.it/macmatem).

The laboratory sessions in the MMLab are designed in order to offer hands-on activities with mathematical machines for classes of students in secondary schools (an average of 1300-1500 Italian secondary students a year come with their mathematics teacher to experience hands-on mathematics laboratory), groups of university students, prospective and practicing school teachers (Bartolini Bussi & Maschietto, 2008). As the Mathematical Machines activities in school classrooms concerns, the MMLab research group organized different long-term teaching experiments in primary and secondary schools (Bartolini Bussi & Pergola, 1996; Bartolini Bussi, 2005; Bartolini Bussi, M. G., Mariotti M. A., Ferri F., 2005, Maschietto & Martignone, 2007).

All the activities quoted above are based on two fundamental components: the idea of the "mathematics laboratory"[1] and the didactical research on the use of tools in the teaching and learning of mathematics (Bartolini Bussi & Mariotti, 2007).

The MMLab researches aim at the development of different activities that should foster, through the use of the mathematical machines, the acquisition of some important mathematical concepts and the emergence of argumentation processes.

In order to implement the studies on MMLab laboratory activities, and to set up new teaching experiments, we consider important to carry out a cognitive analysis of the processes involved in the manipulation of the Mathematical Machines. The aim of our research is identifying Mathematical Machines utilization schemes and the connected exploration processes, providing a first classification. In the paper we shall present the first steps of this new research.

THEORETICAL FRAMEWORK

According to the educational goals that the activities with Mathematical Machines intend to realize, we investigate students cognitive processes involved in exploration of open-ended problems (in particular the problem of identifying the geometrical laws that explain how a machine works), in generation of conjectures and argumentations and in concept formation (for example: the concepts of geometrical transformations, of conic, of central perspective…). First of all, to analyse deeply these processes we propose a classification of Mathematical Machine utilization schemes [2]. This classification is suitable not only for describing the interactions between machines and subjects but also for analysing both their exploration and argumentative processes.

The processes through which a subject interacts with a machine have been studied by Rabardel in cognitive ergonomics: he grounded his research in constructivist epistemologies, primarily in *activity theories*, but also in the Piagetian and post-Piagetian developmental approach to the cognition-action dialectic (Rabardel, 1995; Béguin & Rabardel, 2000).

Rabardel proposed an original approach blending anthropocentric and technocentric approaches: as a matter of fact, in line with activity theory, he conceived the instruments as psychological and social realities and studied the instrument-mediated activity. According to Rabardel (1995) an *instrument* (to be distinguished from the material -or symbolic- object, the *artefact*) is defined as a hybrid entity made up of both artefact-type components and schematic components that are called *utilization schemes*.

> "What we propose to call " utilization scheme" (Rabardel, 1995) is an active structure into which past experiences are incorporated and organized, in such a way that it becomes a reference for interpreting new data" (Béguin & Rabardel, 2000)

An artefact only becomes an instrument through the subject's activity. This long and complex process (named *instrumental genesis*) can be articulated into two coordinated processes: *instrumentalisation*, concerning the individuation and the evolution of the different components of the artefact, drawing on the progressive recognition of its potentialities and constraints; *instrumentation*, concerning the elaboration and development of the utilization schemes (Béguin & Rabardel, 2000).

For the importance of these schemes, for their specificity in interacting with Mathematical Machine and for the limits that this paper has to respect, we focus here on utilization schemes in the case of pantographs.

METHODOLOGY

The method used for investigation was the clinical interview: subjects were asked to explore a machine and to express their thinking process aloud at the same time. In particular, after having explained to the student that the machines to be explored are pantographs for geometric transformations, we asked:
1. To define the mathematical law made locally by the articulated system.
2. In particular, to justify how the machine "forces a point to follow a trajectory or to be transformed according to a given law" and then to prove the existing relationship between the machine properties (structure, working…) and the mathematical law implemented.

The interviews were videotaped and the analysis is mainly based on the transcripts of the interviews. The interviews were analysed with special attention to verbal tracks and hands-on activities in order to detect mental processes developing during the exploration of the machines. Every protocol is analysed in a double perspective: as bearer of new information about possible exploration processes and as evidence for the existence of recurrent schemes.

The subjects were three pre-service teachers, two university students and one young researcher in mathematics. The choice to interview subjects which are familiar with (Euclidean) geometry and with problem-solving has allowed us to collect observations of complete machine exploration: namely, the generation of conjecture about the mathematical law implemented by the machine and, subsequently, argumentation and proof of mathematical statements that can explain the functioning of the machine. Moreover, the subjects were new in working with this environment: in this way we could assume that they did not have an a priori specific knowledge about these machines.

The artefacts selected for this first research are machines concerning geometry, in particular pantographs: for the axial symmetry, for the central symmetry, for the translation, for the homothety and for the rotation. These machines establish a local correspondence between points of limited plan regions connecting them physically by an articulated system; they were built to incorporate some mathematical properties in such a way as to allow the implementation of a geometrical transformation (i.e. axial symmetry, central, translation, homothety, rotation).

CLASSIFICATION OF THE UTILIZATION SCHEMES

In this paper we present the first part of our research that aimed to introduce a classification of utilization schemes observed during the explorations of pantographs for geometrical transformations. The identified utilization schemes were divided into two large families: utilization schemes linked to the components of the articulated system (as the constraints, the measure of rods, the geometrical figures representing a configuration of rods, etc.) and utilization schemes linked to the machine movements. As regards the first family, we have identified the following utilization schemes: the research of fixed points, movable points (with different degrees of freedom), plotter points and straight path; the measure of rods length; the research of geometric figures representing the articulated system or some part of it; the construction of geometric figures that extend the articulated system components; the individuation of relationships between the recognized geometric figures; the analysis of the machine drawings.

As regards the utilization schemes linked to the machine movements [3], we distinguish between the movements aimed at finding particular configurations obtained stopping the action in specific moments and the continuous movements aimed to analyse invariants or changes. We summarize this classification in a table:

Linkage Movement that stops in	**Movements description:**
Generic Configurations	Movement that stops in a configuration which is considered representative of all configurations observed (that does not have "too special" features)
Particular Configurations	Movement that stops in a configuration that presents special features (i.e. right angles, rods positions…)
Limit Configurations	Movement that stops in configurations in which the geometric figures that represent

	the articulated system become degenerate
Limit zones	Movement that stops in the machine limit zones: i.e. the reachable plane points

Linkage Continuous movements	**Movements description:**
Wandering movement	Moving the articulated system randomly, without following a particular trajectory
Bounded movement (For example: Movements by fixing one point or one rod…)	Moving the articulated system, blocking particular points or rods
Guided movement	Moving the articulated system, forcing a point to follow a line or a specific figure
Movement of a particular configuration	Moving the articulated system, maintaining a particular configuration
Movements between limit configurations	Moving the articulated system so that it can successively assume the different "limit Configurations"
Movement of dependence	Moving (in a free, guided or bound way) a particular point and see what another particular point does
Movement in the action zones	Moving the articulated system in a such a way that all the possible parts of the plane are reached

A PROTOCOL

In this paragraph we present the first part of one clinical interview transcripts dealing with the exploration phase (i.e. the beginning of the machine exploration, before the identification of the geometrical transformation made by the machine), where we can identified some of the utilization schemes described in the previous paragraph [4]. The subject of the protocol, Anna, is a pre-service teacher graduated in mathematics and she explored the pantograph of Scheiner (see Fig. 1-2).

Anna: (*she touches a rod which seems to remain blocked*) all motionless!...(*she moves the articulated system*) Ah, no, only a single fixed point … I saw that leads are useful, and then… … (*opening and closing the linkage, she draws lines that converge in the fixed point*) … then (*she turns the machine and she draws again "concentric lines"*)…

She starts controlling which part of the linkage is pivoted to the wood plane (*research of fixed points)* and then, in order to explore the linkage movements, she puts the leads in both plotter holes (*individuation of plotter points*) and draws curves produced by the linkage closing movement (*guided movements that end in a limit configuration*: see Fig. 3)

Anna: I do not see anything then………(*she is looking the motionless machine and the curves drawn*)…(*she moves the linkage and she stops in a generic configuration*) well, this is a parallelogram, I would say… That is… then, parallelogram, and in a vertex there is a lead… (*with the ruler she measures two rods: in the fig. 2 CQ and CP*)… are congruent (*she points them out)*

The analysis of the drawn curves does not seems to help her to discover what transformation the machine makes, therefore she starts an analysis of the linkage structure (*research and individuation of a generic configuration and recognition of particular geometric figures in the linkage structure*): at first she identifies a parallelogram (see Fig.4), and then she focuses on other linkage rods (those parts that do not form the parallelogram). She recognizes the parallelogram without using the ruler (probably the visual perception of congruence has been supported by the previous exploration of movements during which the rods remained parallel). Differently, to discover the other characteristics of the linkage geometric structure, Anna feels the need to *measure the rods length*, so she discovers that there are two congruent rods (CQ and CP).

Anna: … so this (*she looks at the linkage and she uses two fingers to show the "virtual segment" PQ that completes the triangle PQC:* see Fig. 5) is an isosceles triangle

The identification of these congruent rods arouses the construction of a new geometric figure (an isosceles triangle) created completing, with an imaginary segment, the sequence of the congruent rods (*extending and individuation of geometric figures in the linkage structure*).

Anna: but I will not see anything… but it doesn't say anything to me at this moment…… (*she moves the machine, drawing always concentric lines*) well they are always circumferences…(*she is looking at the drawings*) I do not understand if they are or not circumferences …

Also the exploration of linkage characteristics does not seem to help her, for this reason she comes back to the previous strategy: she starts again to draw lines that follow the machines closing movement (*guided movements that end in a limit configuration and analysis of these drawings*), but, as before, she is not aware of the drawn lines characteristics; therefore, not knowing which properties designed curves have, she can not understand how they are transformed by the machines.

Anna: (*she makes a zigzag movement*) well, but it seems to me that they trace the same thing (*she makes the zigzag movement in another area of the paper*)… (*she points the zigzag drawing and she moves away the linkage*)… the leads

> then trace the same, the same image, it seems to me, but I dare say that (*she makes a gesture*: see Fig.6)…that it is reduced in scale.

Anna changes the *guided movements* (zigzag movements) and, this time, the *analysis of the drawings* leads to the recognition of the transformation (the homothety). Therefore it seems that what lets Anna to do the discovery of the transformation incorporated in the machine, is the drawings analysis more than the machine structure; but not all the drawings seem to be successfully: in fact each of them gives only partial information about the transformation. In particular, for Anna is determinant the choice to change the movement (and consequently, the drawing): as a matter of fact in the zigzag lines it can be seen that the correspondent segments are modified, while the angles are not (in the previous drawings these proprieties are "hidden", while it came out the presence of a fixed point).

In conclusion, it is interesting to underline that also in a brief excerpt, it is possible to see the variety, the complexity of their relationships and, in particular, the plot of the different utilization schemes. After the individuation of the schemes, we can make a cognitive analysis of the exploration processes linked to these schemes. For example, we intend to examine closely how (and then why) Anna swings between two different strategies that remain separated (the drawing/analysis of lines and the study of linkage structure). This analysis brings important information for the understanding of subsequent processes: in fact, in the continuation of this protocol, the lack of interweaving of the information acquired through the different utilization schemes used, seems to be an obstacle in the Anna's proof construction (about how the machine incorporates the transformation properties). This part of the research is still in progress, but the first results raise the hypothesis that successful strategies are those that maintain a tension and integration between the analysis of the articulated system proprieties, the drawings and the invariants of the movement.

CONCLUDING REMARKS

The studies on the interaction between a subject and a machine have to take into account an intriguing complexity because several components are involved. From a cognitive point of view and with educational goals, in this paper, we have presented a study to better understand the exploration of some geometrical machines: in particular, we have proposed a first classification of utilization schemes of pantograph for geometrical transformations and we have shown an analysis carried out through this classification. In this analysis we have underlined the importance of the identification of the different schemes in describing the aspects of mathematical machines exploration.

Further researches are needed in two directions. On the one hand, we will study how these schemes are intertwined with the processes involved in conceptualisation, in argumentation and in proving; on the other hand, we will explore the evolution of the utilization schemes and its relationship with argumentation processes and subject's cultural resources.

Moreover, this study will be developed to offer teachers tools that could be efficient to set up activities with educational goals and to intervene in students' interactions with the machines, promoting those processes that are considered relevant for the activities with the mathematical machines.

NOTES

1. "A mathematics laboratory is a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. (…) The mathematics laboratory shows similarities with the concept of Renaissance workshops where apprentices learned by doing and watching what was being done, communicating with one another and with the experts" http://umi.dm.unibo.it/italiano/Didattica/ICME10.pdf.

2. In literature there are not previous cognitive studies of this type on mathematical machines. A classification of utilization schemes of instruments of different nature is proposed in Arzarello et al. (2002) where different modalities of dragging are discussed.

3. In addition to the linkage movements, there are also the movements of the machine wood base (on which the linkage is set): i.e. the rotations of the base that permit to look the machine from other points of view.

4. In these extracts there are not all the utilization schemes identified during our research. For the limit of this article we should not make an example for each of the utilization schemes previously listed.

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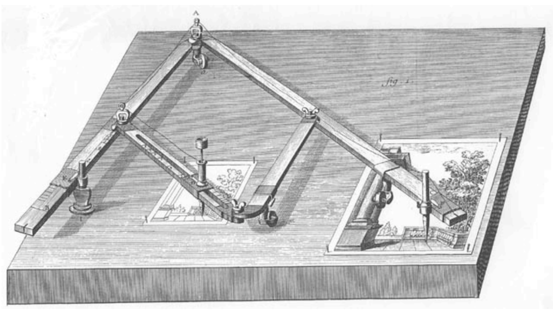


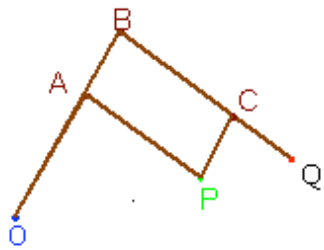
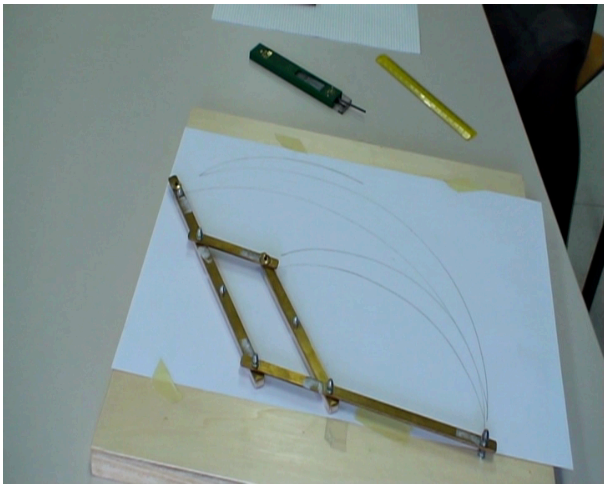


Fig 1: Encyclopédie ou dictionnaire raisonné des sciences, des arts et des metiers (1751-1780)	**Fig 2**: An image from Scheiner pantograph graphic animation: Four bars are pivoted so that they form a parallelogram APCB. The point O is pivoted on the plane. It is possible to prove that the points P, Q and O are in the same line and that P and Q are corresponding in the homothetic transformation of centre O and ratio BO/AO.



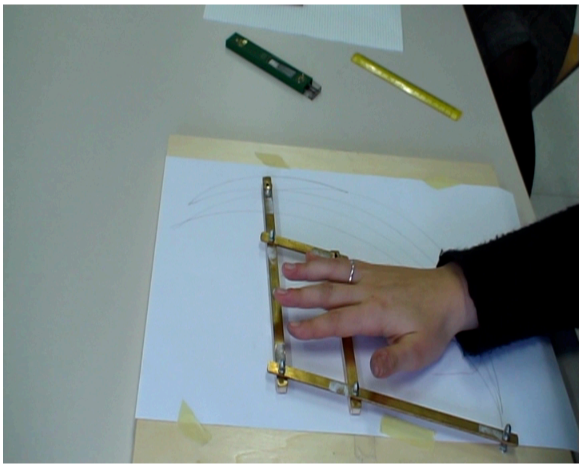
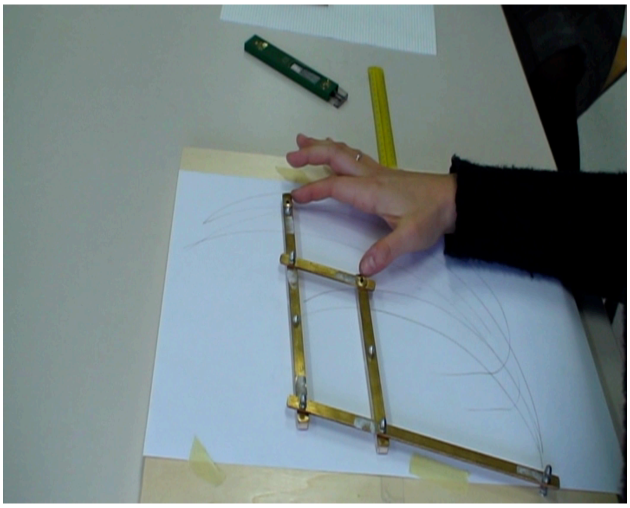


Fig. 3: Anna's drawings	**Fig. 4**: Anna identifies the parallelogram



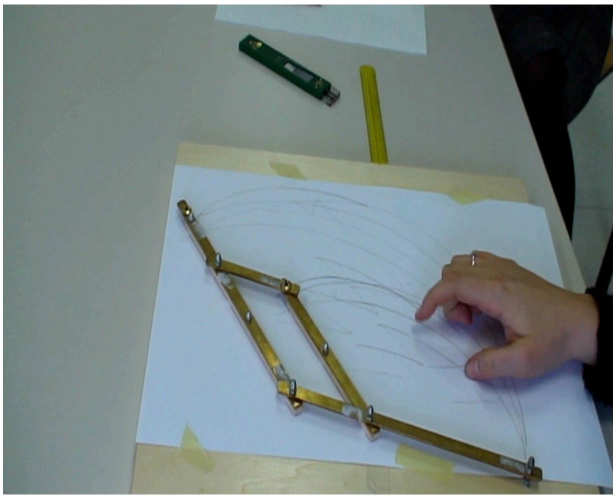


Fig. 5: Anna shows the isosceles triangle	**Fig. 6**: Anna's gesture for indicating the "reduction in scale" of the zigzag lines

MAKING SENSE OF STRUCTURAL ASPECTS OF EQUATIONS BY USING ALGEBRAIC-LIKE FORMALISM

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This paper reports on a design experiment conducted to explore the construction of meanings by 17-year-old students, emerging from their interpretations and uses of algebraic-like formalism. The students worked collaboratively in groups of two or three, using MoPiX, a constructionist computational environment with which they could create concrete entities in the form of Newtonian models by using equations and animate them to link the equations' formalism to its visual representation. Some illustrative examples of two groups of students' work indicate the potential of the activities and tools for expressing and reflecting on the mathematical nature of the available formalism. We particularly focused on the students' engagement in reification processes, i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

INTRODUCTION

In this paper we report on a classroom research [1] aiming to explore 17-year-old students' construction of meanings emerging from the use of algebraic-like formalism in equations used as means to create and animate concrete entities in the form of Newtonian models. The students worked collaboratively in groups of two or three using a constructionist computational environment called "MoPiX" [2], developed at the London Knowledge Lab (http://www.lkl.ac.uk/mopix/) (Winters et al., 2006). MoPiX allows students to construct virtual models consisting of objects whose properties and behaviours are defined and controlled by the equations assigned to them. We primarily focused on how students interpreted and used the available formalism while engaged in *reification* processes (Sfard, 1991), i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

THEORETICAL BACKGROUND

Recognising the meaning of symbols in equations, the ways in which they are related to generalisations integrated within specific equations and also the ways in which a particular arrangement of symbols in an equation expresses a particular meaning, are all fundamental elements to the mathematical and scientific thinking. Research has been showing rather conclusively that the use of symbolic formalisms constitutes an obstacle for many students beginning to study more advanced mathematics (Dubinsky, 2000). Traditional approaches to teaching equations as part of the mathematics of motion or mechanics seem to fail to challenge the students' intuitions since they usually encompass static representations such as tables and graphs which are subsequently converted into equations. Lacking any chance of interacting with the respective representations, students fail to identify meaningful links between the components and relationships in such systems and the extensive use of mathematical expressions (diSessa, 1993). Indeed, students tend to use and manipulate physics equations in a rote manner, without understanding the concepts they covey (Larkin et al., 1980). Sherin (2001) argued that, in order to overcome this obstacle, students need to acquire knowledge elements that he termed *symbolic forms*. The acquisition of *symbolic forms* would help students make connections between an algebraic expression's conceptual content and its structure, which is considered to be crucial for the understanding, meaningful use and construction of physics equations.

In the mathematics education field, the relevant research is mainly based on the distinction between the two major stances that students adopt towards equations: the process stance and the object stance (Kieran, 1992; Sfard, 1991). The process stance is mainly related with a surface "reading" of an equation concentrated into the performance of computational actions, following a sequence of operations (i.e. computing values). In contrast, according to the object stance, an equation can be treated as an object on its own right, which is crucial to the students' development of the so-called *algebraic structure sense* (Hoch and Dreyfus, 2004), i.e. the act of being able to see an algebraic expression as an entity, recognise structures, sub-structures and connections between them, as well as to recognise possible manipulations and choose which of them are useful to perform. This development, linking procedural and structural aspects of equations, has been termed *reification* (Sfard, 1991) and has been considered to underlie the learning of algebra in general.

Recently, students' use and interpretations of symbolic formalism in understanding mathematical and scientific ideas have been studied in relation to the representational infrastructure of new computational environments designed to make the symbolic aspect of equations more accessible and meaningful to children, especially through the use of multiple linked representations (Kaput and Rochelle, 1997). Adopting a broadly constructionist framework (Harel and Papert, 1991), we used a computer environment that is designed to enhance the link between formalism and concrete models, allowing us to study the ways in which the use of formalism, when put in the role of an expression of an action or a construct (a model), can operate as a mathematical representation for constructionist meaning-making. Our central research aim was to study students' construction of meanings emerging from the use of mathematical formalism when engaged in reification processes. We mainly focused on the development of their understanding on the structure of an equation based primarily on the conception of it as a system of connections and relationships between its component parts.

THE COMPUTATIONAL ENVIRONMENT

MoPiX (Winters et al. 2006) constitutes a programmable environment that provides the user the opportunity to construct and animate in a 2d space, models representing phenomena such as collisions and motions. In order to attribute behaviours and properties to the objects taking part in the animations generated, the user assigns to the objects equations that may already exist in the computational environment's Equations Library or equations that she constructs by herself.

Figure 1 shows a red ball performing in the MoPiX environment a combined motion both in the vertical and horizontal axis, leaving a green trace behind. As one may observe, the equations attributed to the object incorporate formal notation symbols (Vx, x, t) as well as programming–natural language utterances (ME, appearance, Circle). However, their main characteristic is that they constitute functions of time, as it is stated by the second argument on the parentheses on their left side. For example, the horizontal motion equations attributed to the ball define the object's: horizontal position at the 0 time instance (1), horizontal position at any time instance (2), the horizontal velocity at the 0 time instance (3), the horizontal velocity at any time instance (4) and the horizontal acceleration at any time instance (5). The MoPiX environment constantly computes the attributes given to the objects in the form of equations and updates the display, generating on the screen the visual effect of an animation.

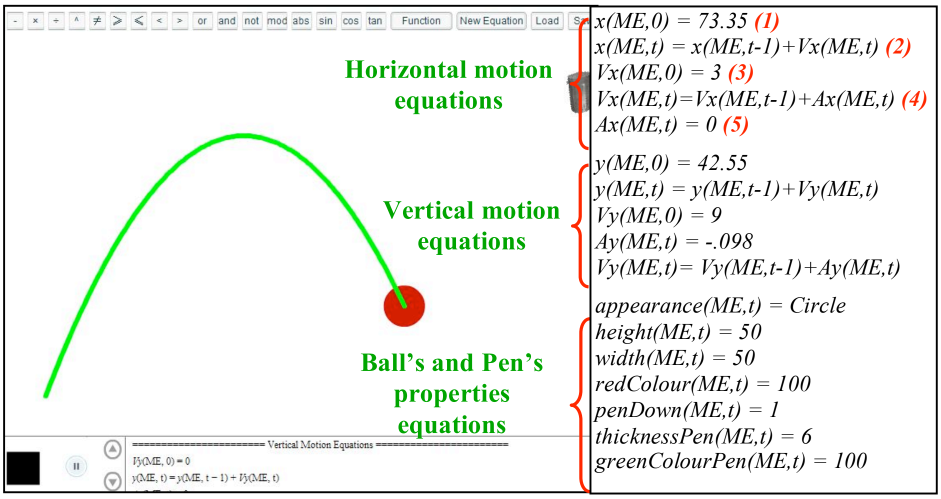

Figure 0. The MoPiX environment

Some specific features of MoPiX, underlying the novel character of the representations provided, may offer students opportunities to further appreciate utilities of the algebraic activity around the use of equations. The first of these features is that MoPiX offers a strong visual image of equations as containers into which numbers, variables and relations can be placed. The meaningful use of the environment may allow students to easily make connections between the structure of an equation and the quantities represented in it. The second feature of MoPiX is that it allows the user to have deep structure access (diSessa, 2000) to the models animated. The equations attributed to the objects and underpin the models' behaviour do not constitute "black boxes", unavailable for inspection or modifications by the user (for a discussion on black and white box approaches see Kynigos 2004). The third feature of MoPiX is that the manipulations performed to a model's symbolic facet (e.g. changing a value or removing an equation from the model) produce a visual result on the Stage, from which students can get meaningful feedback. "Debugging" a flawed animation demands students' engagement in a back and forth process of constructing a model predicting its behaviour, observing the animation generated, identifying the equations that are responsible for the "buggy" behaviour and specifying which and how particular parts need to be fixed.

TASKS

For the first phase of the activities we developed, using exclusively "Library" equations, the "One Red Ball" microworld which consisted of a single red ball performing a combined motion in the vertical and the horizontal axis. The students were asked to execute the model, observe the animation generated, discuss with their teammates and other workgroups the behaviours animated and write down their remarks and observations on a worksheet. In order to provoke discussions regarding the equations' role and stimulate students to start using the equations themselves, we asked them to try to reproduce the red ball's motion. In this process, we encouraged them to interpret and use equations from the "Library", add and remove equations from their objects so as to observe any changes of behaviour and link the equations they used to the behaviours they had previously identified. As we deliberately made the original red ball move rather slowly, near the end of this phase, we expected students to start expressing their personal ideas about their own object's motion (e.g. make it move faster) and thus start editing the model's equations, using the "Equations Editor", so as to describe the new behaviours they might have in mind.

For the second phase of the activities we designed a half–baked microworld (Kynigos 2007), i.e. a microworld that incorporates an interesting idea but it is incomplete by design so as to invite students to deconstruct it, build on its parts, customize and change it. In this case we built a game–like microworld –called "Juggler" (Kynigos 2007)– consisting of three interrelated objects: a red ball and two rackets with which the ball interacted. The ball's behaviour was partially the same as the "One Red Ball's". However, certain equations underpinning its behaviour, did not derive from the environment's "Library" but were created by us. Using the mouse the rackets could be move around and make the ball bounce on them, forcing it to move away in specific ways.

We asked the students to execute the Juggler's model, observe the animation generated and identify the conditions under which each object interacted with each other. The students were encouraged to discuss with their teammates on how they would change the "Juggler" microworld and embed in it their own ideas regarding its behaviour. In the process of changing the half–baked microworld, students were expected to deconstruct the existing model so as to link the behaviours generated on the screen to its equations' formalism and reconstruct the microworld, employing strategies that would depict their ideas about the new model's animated behaviours.

METHOD

The experiment took place in a Secondary Vocational Education school in Athens with one class of eight 12th grade students (17 years old) studying mechanical engineering and two researchers -the one acting also as a teacher- for 25 school hours. Students were divided in groups of two or three. The groups had at their disposal a PC connected to the Internet, the MoPiX manual, translations in Greek of selected equations' symbols and a notebook for expressing their ideas. The adopted methodological approach was based on participant observation of human activities, taking place in real time. The researchers circulated among the teams posing questions, encouraging students to explain their ideas and strategies, asking for refinements and revisions when appropriate and challenging them to express and implement their own ideas. A screen capture software was used so as to record the students' voices and at the same time capture their interactions with the MoPiX environment. Apart from the audio/video recordings the data corpus involved also students' MoPiX models as well as the researchers' field notes. For the analysis we transcribed verbatim the audio recordings of two groups of students for which we had collected detailed data throughout the teaching sequence and also several significant learning incidents from other workgroups. The unit of analysis was the episode, defined as an extract of actions and interactions performed in a continuous period of time around a particular issue. The episodes which are the main means of presenting and discussing the data were selected (a) to involve interactions with the available tool during which the MoPiX equations were used to construct mathematical meaning and (b) to represent clearly aspects of the reification processes emerging from this use.

ANALYSIS AND INTERPRETATIONS

Interpreting existing equations' symbols

In the first phase of the experimentation, the students in their attempt to reproduce the red ball' motion, started interpreting and using equations that already existed in the environment's "Equation Library". The natural language aspect incorporated in the MoPiX formalism was the element that guided their actions. The equations that they chose first to assign to their object were those whose symbols (at least some of them) were close to everyday language utterances and provided them some indication on the kind of the behaviour they described (e.g. the "amIHittingtheGround" symbol). Equations that contained symbols that didn't satisfy the "natural language" criterion (e.g. the "Ax") were simply disregarded.

As they continued their experimentations with MoPiX, the students seemed to gradually abandon the "natural language" criterion and shifted their attention into identifying the meaning of the symbols. The students of Group B for instance came across two "Library" equations that seemed to describe the velocity in the x axis, the "Vx(ME,0)=3" and the "Vx(ME,t)=Vx(ME,t-1) + Ax(ME,t)". Their decision to attribute the second one to their object, so as to define its velocity at any time instance, came as a result of a comparison between the two equations' left parts. Yet again, the students seemed to interpret specific symbols of the equations and completely disregard others (e.g. the "Ax" on the right part).

In a number of subsequent episodes, the same students seem to articulate their understanding not just about particular symbols but also about the whole string of the equation's symbols and the relations among them. In the following excerpt the students of Group B talk about the "x(ME,t)=x(ME,t-1)+Vx(ME,t)" equation.

S1 It *[i.e. x(ME,t)]* is the object *[i.e. "ME"]* in function with time *[i.e. "t"]*.

R2	What does this mean?
S1	*[goes on disregarding the question and points at the x(ME,t-1)]* It's your object *[i.e. "ME"]* in function with time minus 1 *[i.e."t-1"]*.
R2	What does "in function with time" mean? Can you explain it to me?
S1	How much... In every second, for example, how much it moves.
R2	Meaning?
S2	Wait a minute! *[Showing both parts of the equation]* The equation is this one. All of this. It's not just these two *[i.e. the x(ME,t) and the x (ME,t-1)]*.
S1	Minus 1, which means that in every second of your time it subtracts always 1, resulting to something less than the current time. Plus your velocity.

Drawing on his previous experience with the MoPiX equations, S1 starts to independently interpret the equation's symbols moving from left to right. Having interpreted the first two of them, he attempts to also interpret the relationship between them and defines it as the distance that the object has covered in a second of time. S2, who understands the kind of correlation S1 has made, intervenes and stresses the fact that he hasn't taken into account all the symbols in the equation. S1, who up to that point disregarded the "Vx(ME,t)" on the right part, takes an overall view of the equation and interprets it not by merely referring to the comprising symbols but also by referring to the connection between them. It is noticeable that at this point the students' actions demonstrate an emerging awareness of the equation's structure as a system of connections and relationships between component parts.

Variables and numerical values to control motion animations

As students gained familiarity with the MoPiX formalism, they started expressing their own personal ideas about the ways their objects should move. In order to put into effect those ideas, the students initially modified the existing equations' symbols and left the structure intact. One of the main elements that they often altered was the equations' arithmetic values. The students of Group B for instance attributed to their object the "Vy(ME,0)=0" equation which prescribed the object's y axis initial velocity to be 0. The observation of the animation triggered the implementation of a series of changes to the equation's arithmetic values starting with the conversion of the "0" on the right part into "3". The successive changes of the arithmetic value on the equation's right part didn't cause the object to constantly move since the equation referred just to the initial velocity. To make the velocity for "all the next time instances to come" to be "3", the students replaced the "0" on the left part (i.e. an arithmetic value) with "t" (i.e. a variable).

S2	Do we need a symbol for this?
R2	Do we need a symbol? It's a good question. How do you plan to express it?
S2	With symbols we usually express something that we can't describe accurately.

S1 Plus… t. *[He writes down Vy(ME,t)=3]. [Showing the "t"]* So, when I look at this symbol

S2 I'll know it represents the infinity.

We suggest that the students relocated their focus from just replacing specific values, which indicates a process stance to equations, into forming functional relationships. The fact that they were involved in a process of recognizing which manipulations were possible and at the same time useful to perform so as to express their idea, indicates a implicit focus on the structure of the equations. Furthermore, the statements concerning the use of symbols to express "something that we can't describe accurately" seems to constitute an indication of a progressive acquisition of algebraic structure sense through "mixed cues" (Arcavi, 1994) (i.e. interpreting symbols as invitations for some kind of action while working with them).

Relating different objects' behaviours by constructing new equations

The next episode describes how the Group A students, in the course of changing the "Juggler" microworld, didn't just use or edit existing equations but constructed from scratch two new ones. The idea they wanted to bring into effect was to "make a ball on the Stage change its colour according to an ellipse's position". Knowing that there was no such equation in the "Library", they started talking about how they would correlate those two objects using the Y coordinates.

S1 When it *[i.e. the ball]* is situated in a Y below the Y of this one *[i.e. the ellipse]* for example.

R1 I'm thinking… Will the ball know when it is below or above the ellipse?

S2 That's what we will define. We will define the Ys.

S1 This. The: "I am below now". How will we write this?

S2 Using the Ys. Using the Ys. The Ys. That is: when its Y is 401, it is red. When the Y is something less than 400, it's green!

Having conceptualized the effect they would like the new equation to have, the students in the above excerpt decide about two distinct elements regarding the equation under construction: its content (i.e. the symbols) and its structure (i.e. the signs between the symbols). Subsequently, encountering the fact that there was no in-built MoPiX symbol to express the idea of an object becoming green under certain conditions, the students came to invent a new symbol. The "gineprasino" (i.e. "become green" in Greek) symbol was decided to represent a varying quantity taking two distinct values (1 and 0 according to if the ball was below the ellipse or not). To represent the ball's position they chose to use its Y coordinate in terms of a quantity varying over time (i.e. "y(ME,t)"). However, to represent the ellipse's position they chose to use its Y coordinate in terms of the constant arithmetic value corresponding to the object's at that time position on the Stage (i.e. "274"). Adding a "less than" sign in between, the equation eventually developed was the "gineprasino(ME,t)=y(ME,t)≤274".

Unexpectedly, this equation didn't cause the ball to become green since it described solely the event to which the ball would respond (being below the ellipse) and not the ball's exact behaviour after the event would have occurred (change its colour). To overcome this obstacle, the students decided to construct another equation in which they tried to find out ways to integrate the "gineprasino" variable. A "Library" equation which explains what happens to a ball's velocity when it hits on one of the Stage's sides and the way in which a variable similar to the "gineprasino" was incorporated in it, led students to duplicate this equation's structure, eliminate any content and use it as a template to designate what happens to the ball's colour when it is below the ellipse. The second equation encompassed in-built MoPiX symbols (the "greenColour"), the "gineprasino" variable in two different forms (not(gineprasino) and gineprasino) and numerical values (0 and 100) to express the percentage of the green colour the ball would contain in each case (i.e. the ball being above and below the ellipse). Thus, the second equation developed was the: "greenColour(ME,t) = not(gineprasino(ME,t))×0 + gineprasino(ME,t)×100".

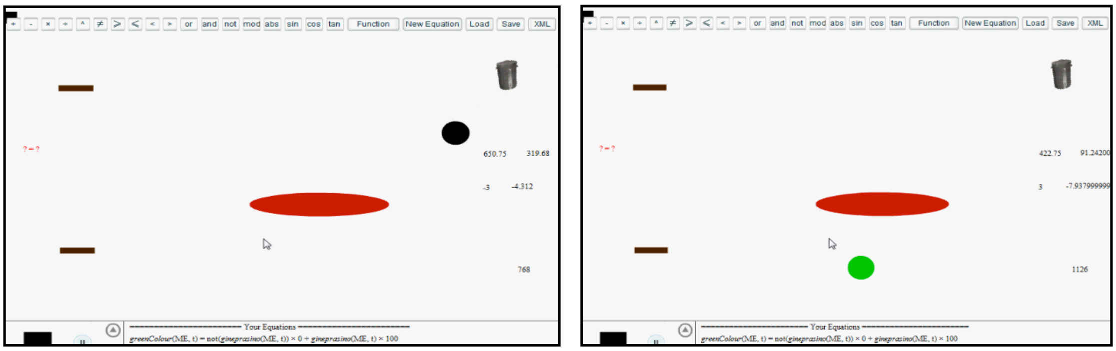

Figure 2: The ball's different percentage of green colour according to its Y position

The above episode contains many interesting events that indicate the existence of a qualitative transformation of the students' mathematical experience in reifying equations that emerged through their interaction with the available tools.

While building the first equation the students got engaged in processes such as inventing and naming variables, relating symbols with mathematical systems (i.e. the XY coordinate system) and manipulating inequality symbols to relate arithmetic values and variables. However, in building the second equation, the meaning generation evolved to include the students' view of equations as objects. The students extracted mathematical meaning from an equation that seemed to describe a behaviour similar to the one they intended to attribute to their ball. Conceptualizing a mapping between the ideas behind the two equations, the students duplicated the similar equation's structure and inserted new terms so as to define a completely novel behaviour for their object. This is a clear indication that they recognised the existence of structures external to the symbols themselves and used them as landmarks to navigate the second equation's construction process.

The manipulation of the second equation's new terms reveals further their developing structural approach to equations. By inserting in the second equation the the "gineprasino" variable which was introduced in the first one and providing it new forms (i.e. not(gineprasino)), the students seem to have conceptualised the first equation as a mathematical object which it could be used means to encode structure and meaning in the second equation. We think that this reflects a kind of mathematical thinking that has a great deal to do with developing a good algebraic structural sense accompanied with the acquisition of a functional outlook to equations as objects which is a warranty of relational understanding.

CONCLUDING REMARKS

Our purpose in this paper was to illustrate a particular approach to studying the student's construction of meanings for structural aspects of equations, emerging from the use of novel algebraic-like formalism. In the first part of the results, an initial icon-driven conceptualisation of the MoPiX equations seemed to have been leading students towards the development of criteria for an isolated interpretation of the MoPiX equations' symbols. As soon as the students became familiar with testing their models and observing the animations generated on the "Stage", their interactions with the computer environment became strongly associated with the editing of the existing equations' content. As expressed in the second part of the results, the editing of equations revealed a subtle shift from a process-oriented view to equations into an object- oriented one as well as a progressive development of algebraic structure sense. In the last part of the results, students' previous experience with the MoPiX tools seemed to become part of their repertoire, allowing them to construct new equations following specific structural rules, invent variables and specify their values, and use the equations as objects to represent variables in other equations. Concluding, we suggest that in the present study reifying an equation was not a one–way process of understanding hierarchically–structured mathematical concepts but a dynamic process of meaning–making, webbed by the available representational infrastructure and the ways by which students drew upon and reconstructed it to make mathematical sense.

NOTES

1. The research took place in the frame of the project "ReMath" (Representing Mathematics with Digital Media), European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2008 (http://remath.cti.gr)

2. "MoPiX" was developed at London Knowledge Lab (LKL) by K. Kahn, N. Winters, D. Nikolic, C. Morgan and J. Alshwaikh.

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COLLABORATIVE DESIGN OF MATHEMATICAL ACTIVITIES FOR LEARNING IN AN OUTDOOR SETTING

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In recent years, teaching mathematics in an outdoor setting has become popular among teachers, as it seems to offer alternative ways to motivate children's learning. These new learning possibilities pose crucial questions regarding the nature of how mathematical activities should be designed for outdoors settings. In this paper we describe our current work related to the design and implementation of mathematical activities in this particular environment in which a specific mathematical content was used as the central component in the design. We illustrate our collaborative design approach and the results from observations of two activities. Our initial results provide us with valuable insights that can help to better understand how to design and implement this kind of educational activities.

INTRODUCTION

A recent trend in Swedish elementary schools is an increasing interest to teach mathematics in an outdoor setting. Teachers believe that this particular approach motivates the children more than solving problems in textbooks, thus offering new ways to introduce and work with mathematical concepts (Lövgren, 2007). Teaching mathematics in an outdoor setting usually refers to school children solving practical problems using whichever forms of mathematics they find appropriate (Molander, Hedberg, Bucht, Wejdmark, Lättman-Mash, 2007). The approach presented in this article is somewhat different. The paper describes our initial efforts with regard to an ongoing project in which a specific mathematical content within the field of geometry was used as the central component in the design of mathematical activities in an outdoor setting.

Our project involves a development team consisting of schoolteachers, university teachers and researchers, who collaborate to develop mathematical activities with the purpose of supporting students' processes of learning. The mathematical activity described in this paper was developed during a period of eight months, counting from the first meeting of the development team and until the completion of the activities involving students. The methodological approach used for developing the mathematical activity will be the central focus of our discussions.

Even if outdoors teaching of mathematics has got an increasing interest among teachers and teacher educators in recent years, we found few published materials with reference to outdoor environments in the research field of mathematics education. For instance, we found no results when searching on *outdoor, outdoors* or *embodied* in titles or keywords in *Educational Studies in Mathematics, Journal for research in Mathematics Education* and *The Journal of Mathematical Behaviour*. When we searched on the term *physical,* some results showed up. However, in a brief check on research methodologies adopted in these studies, no one was centred on an outdoor activity.

Against this background, the current (ongoing) project aims at investigating different possibilities to support students' processes of learning by designing mathematical activities for an outdoor setting. This approach does not aim at replacing traditional mathematics teaching. It should rather be interpreted as a complementary method to be used at the discretion of the mathematics teacher in combination with other teaching methods. In this paper, we particularly aim at discussing our method of design in connection to the principles of Design experiments (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). Throughout the discussions presented in this paper, special attention is paid to the constitution and the working conditions of the development team.

The rest of paper is organized as follows; in the next section we present the mathematical tasks that guided our design and activity while the subsequent section gives a brief overview on the concept of design experiments. The preceding sections illustrate the results from observations of two activities followed by discussions on the notions of group and individual mathematical understanding and practices. The last two sections conclude this article by providing a description of current and coming directions of our work together with a discussion about future challenges.

DEVELOPMENT OF ACTIVITIES

In this section we describe, both the content of the proposed activities as well as the approach taken while designing the different tasks. The driving force in the design process has been experience-based suggestions from the schoolteachers. Each meeting of the development team has involved four to six teachers and two to three university researchers. The first meeting of the development team focused on identifying mathematical content and learning objectives for an outdoor activity suitable for beginners at lower secondary school. We soon agreed to focus on geometry. Aspects that were discussed dealt with the problems students have on understanding geometrical concepts such as area and perimeter. An early idea was to produce a series of activities showing progression from length to area and then to volume, using physical objects close to the school yard. The university representatives suggested utilizing non-standard measurements (sticks, steps and squares) to be used in relation to triangles, rectangles and polygons defined by trees or within the school soccer field. The school teachers instead suggested to focus on four aspects of the selected domain, namely the following learning objectives; comparison of figures, making own estimates, constructing figures with given measures and, specifically, discovering that a doubling of lengths makes the area four times larger.

It was decided that the university teachers should work on designing a task incorporating as many as possible of the agreed suggestions and present it to the whole group after the summer 2007. The proposed mathematical task, as described in figure 1, aimed at having the students construct the following sequence of figures using ropes and metal hooks to be fastened in the ground.

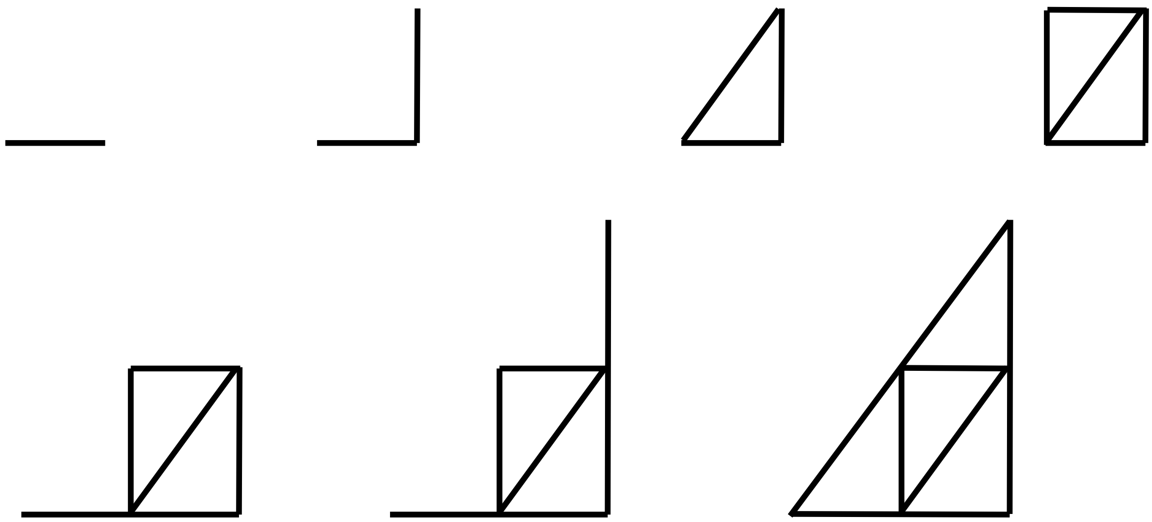

Figure 1: Intended sequence of figures to be constructed by the students.

Shortly after the summer, Växjö University hosted Professor Matthias Ludwig from *Pädagogische Hochschule Weingarten* in Germany, who offered to give two one-hour lectures at our department. One of these discussed outdoor geometrical tasks and tools used in connection with the tasks. Inspired by his lecture we decided to suggest construction of two tools; one for producing a right angle and one for measuring arbitrary angles, both based on making judgments by eyesight. The planned right angle tool consisted of a wooden square with markers at the middle of each side, as shown to the left in Figure 2.

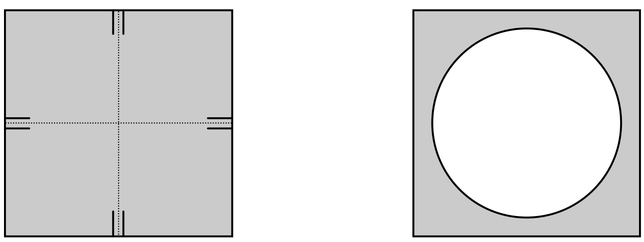

Figure 2: Ludwig's tool to the left, our tool to the right.

The woodwork teacher at the school prepared a number of square boards and also prepared a number of round boards intended for use in another activity. The square shaped tool could also be used to represent a square meter since its side was exactly one meter. However, we identified several disadvantages of this tool with respect to the intended task: it could not be used while placed on the ground, it was quite heavy, and the handling required several people operating close to the tool. We later chose the tool shown to the right in the figure above, which was actually what was left over after the round boards had been cut out. This second tool had several advantages. It could be used while placed directly on the ground, it was easy to carry due to the hole in the middle, and could be used at a distance. The right angle was aimed at the sides of the tool.

In the first proposal, the lengths for the catheti (that were to be doubled during the task) were 3 meters and 4 meters. In the construction, metal hooks and flag lines were used. While trying out the task on the (grass-covered) school yard we all agreed that larger measures were needed, to give the students a better overview of the construction and to give them reason to move within the figure. The first suggestion was to double the lengths to 6 meters and 8 meters, but we also agreed to avoid an exact measure for the hypotenuse and ended up choosing 5 meters and 8 meters as lengths for the catheti.

The task was communicated to the students through written instructions on paper. The first page of the instructions described the tools the students were supposed to bring to the school yard (3 flag lines, 6 metal hooks, roll-out length measure, right-angle tool, paper and pen). Three separate tasks were described on the following three pages.

Each task was divided into three subtasks in the same way (construct a figure, determine perimeter, determine area). This was done for several reasons. Since the students were not used to this kind of activity, we wanted to restrict the content in each subtask. We also wanted to encourage the students to discuss their conclusions on each subtask as a group, especially to verify that the construction was made according to the descriptions as we suspected that they otherwise might focus only on calculations. Also, since the written instructions were not supported by figures, we found it reasonable to restrict each subtask in order not to make it too difficult for the students to interpret the task. Our aim was to let the students work on the tasks without the support from the teacher; thereby inviting them to take on different roles and take more own initiatives than they were used to in their usual mathematics classroom. Another important aspect was that the tasks should allow for applying different solution strategies, such as measuring, calculation, and comparison.

DESIGN EXPERIMENTS

The methodology used in this project is founded on the principles of Design experiments (Cobb et al., 2003). Cobb and colleagues (2003) summarize Design experiments (DE) in five crosscutting features. The first feature, *develop theories*, concerns understanding processes of learning and the means that are designed to support that learning. The second feature, which concerns *control*, may be seen as the focus of the current project: "The intent is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them" (Cobb et al., 2003, p. 10). To develop theories about learning processes, and to try to exert control of such processes, implies the need for *prospective* and *reflective* analyses. Prospective and reflective work is the third feature of DE. On the prospective side, our designs have been implemented with a hypothesized learning process in mind. The activity has been carried out with students and the following reflective work has been based on observations of students' actions. The prospective and reflective aspects come together in a fourth characteristic of DE, *iterative design.* Iterations are carried out with the modification and development of explaining learning and the means of supporting learning. The project so far has included only two iterations which have been based on informal observations with a rather weak theoretical base. Our strategy has been to let the preliminary informal observations guide us toward relevant learning theories to support later iterations. The fifth feature refers to the *pragmatic roots* of DE. As school teachers take active part in the design process, we feel confident that the activities are relevant for teachers' practice.

OBSERVATIONS FROM TWO ACTIVITIES

Two activities involving students have been carried out in the project. The two activities included two different groups of four students (14-15 years old). The activities were neither videotaped nor audiotaped. Instead, two researchers and two teachers observed the activities. The researchers were the same both times.

During the activities, the students were very eager to start working with the lines and hooks. We feel that the division of each task into subtasks made it possible for them to interpret the subtask while arranging lines and hooks. On a few occasions, when they were getting lost in the construction, we had to intervene and ask them to read the instructions again. We also observed that some of the students had problems handling the instruction papers. These problems concern locating and returning to the instructions after they have been left on the ground, as well as documenting answers to the questions.

One specific observation concerned the change in social behaviour. One of the teachers commented on a female student who was busy constructing sides by pulling flag lines:

> Look at her. She seldom takes initiatives in the classroom; she is very quiet and rarely shows interest. Here she is, pulling flag lines, talking to her classmates and really enjoying what she is doing.

Another notable observation can be seen as relating to gender issues. In a group of two boys and two girls, the boys were trying to solve the problem of extending the catheti, seemingly ignoring the girls. As the boys got stuck, one of the girls walked up to the (female) teacher and whispered her solution. The teacher encouraged her to talk to the boys, and the whole group ended up producing the intended construction.

One specific topic of discussion concerning mathematics emerged in our follow-up meetings. To recall, one of our intention with the design was to encourage different solution strategies, such as measuring, calculation and comparison. What was noticed however, was that measuring took a rather dominant role in the activity. Moreover, since the students were not familiar with the Pythagorean Theorem we did not expect them to calculate the hypotenuse of the first triangle, in order to determine its perimeter. However, when the students were asked to determine the perimeter of the larger triangle, i.e. after the catheti of the first triangle being doubled, they also now measured the hypotenuse. None of the students reflected on or argued that also the hypotenuse was doubled. The students did not even reflect on this after the three sides were measured. The data they used for determining the perimeter was the measured data.

During the first activity, the students quickly turned to calculating the area of the larger triangle by the rule; base times altitude divided by two. No attempt was made to compare the larger triangle with the smaller triangle, even if the construction supported looking four smaller triangles within the larger (see Figure 1). In the instructions for the second activity, we therefore explicitly asked the students if they could find out from the constructions any relation between the area of the larger triangle and the smaller triangle. After some discussion and guidance the students at least articulated that the area of the larger triangle was four times the area of the first triangle. However, we were not comfortable that the activity did not by itself provoke the students to involve principles and relations in their discussions.

We observed that the students solved the tasks rather pragmatically and routinely, in terms of measuring and applying rules for calculation. However, we do not have evidence that the students' behaviour depended on conceptual limitations. In the follow-up discussions within the development team we identified possible explanations in terms of the design of the activity and the students' history of being part of a certain educational system. Therefore, to develop the activity and to understand students' actions and potential, we have reached a point where we find it necessary to deepen the theoretical approach of our work, taking into account analytical constructs on several levels of interaction. In the next section we describe principles of the emergent perspective (Cobb et al., 2001), which we find suitable for our purposes.

CONCEPTUALIZING GROUP AND INDIVIDUAL MATHEMATICAL UNDERSTANDING

In Cobb, Stephan, McClain and Gravemeijer (2001) terms, the evolution of mathematical learning in classrooms constitutes of *social* as well as *psychological* structures of behaviour and reasoning. Within the social structure, they identify three analytical categories: *Classroom social norms, Sociomathematical norms* and *Classroom mathematical practices*. Examples of Classroom social norms can be for instance; that students collaborate to solve problems, that meaningful activity is valued more than correct answers, and that partners should reach consensus as they work on activities. With reference to our observations, Classroom social norms may have been in play when the quiet girl had to be encouraged by her teacher to communicate with her team members. *Sociomathematical* norms are defined as social constructs specific to mathematics. These are the norms in play when explanations and justifications are made acceptable (Hershkowitz and Schwarz, 1999). When applying the analytical construct of *classroom mathematical practices* the analytical lens is closer to a certain instructional activities. It concerns regularities of the collective engagement in a specific situation in terms of symbolizing, arguing and validating.

A student may experience a study activity in different ways, as compared to the teacher's and to other students' interpretations (Wistedt, 1987; Iversen and Nilsson, 2007). The psychological perspective concerns the nature of individual students' reasoning. It brings attention to the diversity in students' ways of interpreting and acting in mathematical activities (Cobb et al., 2001).

It is crucial to understand that the relation between the social and the psychological perspective is considered to be reflexive (Cobb et al., 2001): "…neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective" (p. 122).

An implicit assumption of the current project has been that an unfamiliar teaching arrangement might encourage students to act beyond previously established Classroom social and Sociomathematical norms, with the possibility that these new actions may be more mathematically productive than their correlates of ordinary classrooms. The initial results of our observations, specifically the two separate incidents involving girls, support this assumption.

THE ORGANIZATION OF MATHEMATICAL PRACTICES

Weber, Maher, Powell, and Lee (2008) summarize some important ways in which discussions may establish opportunities for the learning of mathematics. Discussion can objectify students' experiences, thereby making these experiences the subject of analysis, encourage students to take a more reflective stance on their mathematical reasoning, require students to consolidate their thinking by verbalizing their thoughts, and help students learn to communicate mathematically and participate in a wider range of mathematical argumentation. Weber et al., (2008) also contend that group discussion can facilitate learning by inviting students to be explicit both about the ways in which they make new claims from previously established facts and about the standards they are using in deciding whether an argument is acceptable. Challenges from classmates can encourage students to debate whether a particular method of argumentation is appropriate and provide students with the opportunity either to justify their methods when their reasoning is sound or revise or abandon their methods when their reasoning is flawed.

In the organization of group discussions, Cobb et al., (2001) distinguish between three specific structures: taken-as-shared purposes, taken-as-shared ways of reasoning with tools and symbols, and taken-as-shared forms of mathematical argumentation. A taken-as-shared purpose is what the students and the teachers are trying to achieve together mathematically. The second structure is concerned with the ways in which tools and symbols are used and given taken-as-shared meanings. To account for taken-as-shared forms of argumentation Toulmin's (1969) analytical model of argumentation has proven useful (Cobb et al., 2001). According to Toulmin (1969), an argumentation consists of at least three core components: *the claim, the data*, and *the warrant*. When a speaker makes a claim he or she may be challenged to present evidence or data to support that claim. The data typically consist of facts that lead to the conclusion that is made. If a listener does not understand why the data justify the conclusion that was drawn she may challenge the presenter to clarify why the data led to the conclusion. When this type of challenge is made and a presenter clarifies the role of the data in making her claim the presenter is providing a warrant. A warrant can of course be questioned, thus obligating the presenter backing up the warrant.

DISCUSSION ON OUR METHOD OF DESIGN

Our choice of method has been influenced by the constitution and working conditions of the development team. The main focus has been on collaborative development of the mathematical activity. The project emphasizes the potential benefits of collaborative development in close interaction with stakeholders. There has been a very open climate of discussion where teachers' knowledge and experiences have been given equal attention as input from the researchers. The teachers have been very active providing ideas and reacting on suggestions from the researchers, both during physical meetings and through e-mail communication. We argue that this way of collaboration differs from the approach usually used by DE practitioners. In DE, theories are usually introduced in early stage of the design process (diSessa & Cobb, 2004). From the observations of two activities, we have been identified a need for supporting theories. The interpretative frameworks outlined above will enable us to strengthen our design and to better understand our observations. However, we have found it fruitful to use an experienced based approach. No theories have been explicitly communicated during the initial work of the development team. Particularly, we believe that introducing abstract theories early in the discussions would have reduced the teachers' interest and possibilities to communicate empirically grounded ideas, thereby limiting the pragmatic root of the project. Our approach may therefore serve as a reasonable model for others, who wish to engage in collaborative activities in order to enhance school teaching. On account of this, we suggest that researchers in collaboration with teachers should take seriously the role of theories, particularly when to introduce them in the project at hand.

We suggest a balance between theories and practice, where practice takes on a rather dominant role in the early work. As the project and iterations proceed, the role of theories may be increased in order to enhance control of the learning activity. The analytical categories argued by Cobb et al., (2001), and Toulmin's (1969) model of argumentation, offer instruments both for supporting the design process and for serving as tools for analysis of observed actions.

Finally, one can question the validity of our approach in relation to the pedagogical implementation and learning outcomes of these activities but the main point here is not assess the effectiveness of the learning materials neither the mathematical content but instead to explore how to design and organize the flow of pedagogical activities in an outdoor learning setting. Our initial impressions indicate that this kind of learning activities seem to encourage discussions and new collaboration patterns, thus promoting deeper understanding among students. Therefore, we believe that a major challenge for the mathematics education community is to create new possibilities for learners to understand complex mathematical concepts, as well as to develop new analytical tools and theories in order to facilitate our understanding on how learning takes place under these new circumstances.

FUTURE EFFORTS

Based on the discussions presented in this paper, the following suggestions appear to be relevant for the design of the next iteration. The design of the next activity should take into consideration how:

- collective understanding can be provoked by encouraging students to make claims and be explicit about the warrants on which the claims rest,
- collective discussion can capitalize on individual variations (implying that the activity should encourage a variation in reasoning and solution strategies),
- norms and structures of mathematical practices may support or limit students' behaviour.

The last aspect specifically refers to the observation of how measuring took on a rather dominant role in the activities, narrowing the students' conceptual structures. On account of these guidelines we suggest to follow up the described activity with a second activity, where the students are not allowed to use a measuring tool. Instead they start with a triangle with given perimeter and given area and whose sides are not known. The triangle will be marked with flag lines and the students will be asked to continue the construction of the same pattern as in the previous construction and will be asked to determine the perimeter and the area of the larger triangle. We conjecture that such a setup will provoke the students to reflect on conceptual aspects, by comparing features of the triangles. Another suggestion is to let the students choose their own measures and to let them construct a triangle which will be extended to a rectangle, with the aim that they will discover the connection between the areas of the two figures.

An obvious next step of the project is to investigate how the described outdoor activity can be followed up in the regular classroom. Earlier mentioned shortcomings concerning students' documentation may be overcome by using mobile technologies. According to Spikol and Milrad (2008), mobile technologies offer the potential for a new phase in the evolution of technology-enhanced learning, marked by a continuity of the learning experience across different learning contexts. In particular, we propose to let students use mobile technology in order both to communicate the tasks and to support the documentation of their solutions. Moreover, offering the students possibilities to videotape and taking pictures during the activity will support them in recalling and sharing experiences when they return to their regular classroom. Apart from the field of geometry, we believe that interesting applications may be developed in additional fields such as arithmetic and statistics, and even in algebra and functions. Our ambition is to invite students from the teacher training program at our university, so they can participate in widening our design approach to the above mentioned fields.

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THE UTILIZATION OF MATHEMATICS TEXTBOOKS AS INSTRUMENTS FOR LEARNING

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The mathematics textbook is one of the most important resources for teaching and learning mathematics. Whereas a number of studies have examined the use of mathematics textbooks by teachers there is a dearth of research into the use of mathematics textbooks by students. In this paper results of an empirical investigation of the use of mathematics textbooks by students as an instrument for learning mathematics are presented. Firstly, a method to collect data on student's use of mathematics textbooks is introduced. It is explicated, that this method is capable to explore the actual use of the mathematics textbook by students, and a way of recording the use of the mathematics textbook whenever and wherever students use it. Secondly, results from the study are presented. The results outlined in this paper focus on typical self-directed uses of the mathematics textbook by students.

INTRODUCTION

Research in mathematics education has been concerned with the role of new technologies in the teaching and learning of mathematics from the very beginning computers and information technologies entered the mathematics classroom. In the first ICMI study the computer is even considered to be a new dimension in the mathematics classroom: "We now have a triangle, student-teacher-computer, where previously only a dual relationship existed" (Churchhouse et al., 1984). But, this perspective disregards the fact that tools have always been incorporated in teaching and learning mathematics and thus the relationship in the mathematics classroom has never actually been dual. The mathematics textbook was and still is considered to be one of the most important tools in this context. According to Howson, new technologies have not affected its outstanding role: "despite the obvious powers of the new technology it must be accepted that its role in the vast majority of the world's classrooms pales into insignificance when compared with that of textbooks and other written materials." (Howson, 1995)

Valverde et al. (2002) believe that the structure of mathematics textbooks is likely to have an impact on actual classroom instruction. They argue, that the form and structure of textbooks advance a distinct pedagogical model and thus embody a plan for the particular succession of educational opportunities (cf. Valverde et al., 2002). The pedagogical model only becomes effective when the textbook is actually used. Therefore, mathematics textbooks should not be a subject to analysis detached from its use. It is an interactive part within the activities of teaching and learning mathematics In order to develop a better understanding of the role of the mathematics textbooks within the activities of teaching and learning mathematics an activity theoretical model was developed (Rezat, 2006):

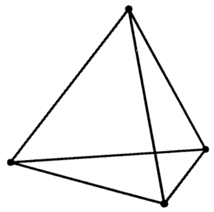


Fig. 1: Tetrahedron model of textbook use

This model is based on the fundamental model of didactical system: the ternary relationship between student, teacher, and mathematics (Chevallard, 1985). The mathematics textbook is implemented as an instrument at all three sides of the triangle: teachers use textbooks in the lesson and to prepare their lessons, by using the textbook in the lesson teachers also mediate textbook use to students, and finally students learn from textbooks. Thus, each triangle of the tetrahedron-model represents an activity system on its own. From an ergonomic perspective it is argued that artefacts have an impact on these activities, because on the one hand they offer particular ways of utilization and on the other hand the modalities of the artefacts impose constraints on their users (cf. Rabardel, 1995, 2002). Thus, the mathematics textbook has an impact on the activity of learning mathematics as a whole that is represented by the didactical triangle on the bottom of the tetrahedron.

Whereas a number of studies have examined the role of new technologies in terms of tool use (cf. Lerman, 2006) the role of the mathematics textbook as an instrument for teaching and learning has not gained much attention. So far, a number of studies have examined the use of mathematics textbooks by teachers (e.g. Bromme & Hömberg, 1981; Haggarty & Pepin, 2002; Hopf, 1980; Johansson, 2006; Pepin & Haggarty, 2001; Remillard, 2005; Woodward & Elliott, 1990) whereas there is a dearth of research into the use of mathematics textbooks by students (Love & Pimm, 1996). This is striking, because as pointed out by Kang and Kilpatrick (1992), textbook authors regard the student as the main reader of the textbook.

In order to develop a better understanding of the impact that textbooks have on learning mathematics a qualitative investigation was carried out in two German secondary schools that focused on how students use their mathematics textbooks.

METHOD AND RESEARCH DESIGN

The difficulty of obtaining data on students working from textbooks is one reason that Love and Pimm (1996) put forward in order to explain the dearth of research into student's use of texts. Therefore, developing an appropriate methodology to collect data on student's use of mathematics textbooks can be regarded as a major issue in this field.

First of all, the method of data-collection has to be in line with the situation of textbook use. In Germany, schools either provide mathematics textbooks to students for one year or students buy the books. Accordingly, students have access to their mathematics textbook at school and at home. From previous research there is evidence, that German teachers rely heavily on the textbook in the preparation of lessons and also during lessons. (Bromme & Hömberg, 1981; Hopf, 1980; Pepin & Haggarty, 2001).

The method to collect data on student's use of mathematics textbooks was developed within the framework of the activity theoretical model of textbook use. According to this model the use of mathematics textbooks is situated within an activity system constituted by the student, the teacher, the mathematics textbook, and mathematics itself. First of all, this implies that a method to investigate the use of mathematics textbooks by students has to take all four vertices of the tetrahedron-model into consideration.

In addition, three criteria were established for an appropriate methodology to collect data on student's use of mathematics textbooks:

1. The actual use of the mathematics textbook should be recorded in detail.
2. Biases caused by the researcher, by the situation or by social desirability should be minimized.
3. The use of the textbook should be recorded at any time and any place it is used.

Criterion 1 leads to the rejection of quantitative methods and of methods that are likely to reveal only verbalized uses of the textbook, e.g. interviews. Experimental settings and artificial situations are refused due to criterion 2. Approaches that are solely based on observation are discarded because of criterion 3.

The methodological framework that was developed according to the three criteria combines observation and a special type of questioning. First of all, the students were asked to highlight every part they used in the textbook. Additionally, they were asked to explain the reason why they used the part they highlighted in a small booklet by completing the sentence "I used the part I highlighted in the book, because …". By assigning more than one comment to a highlighted book section the reuse of book sections becomes apparent. This method of data-collection was developed in order to get the most precise information about what the students actually use and why they use it by keeping the situation of textbook use as natural as possible. Nevertheless, highlighting sections in a textbook is not the natural way to use the textbooks and therefore a bias on the data cannot be totally excluded.

Provided that the students take their task seriously, this method enables to collect data on the use of the textbooks whenever and wherever students use it and therefore meets criterion 3.

In addition, the lessons were observed and field notes were taken. On the one hand the overall structure of the lesson was recorded in the field notes using a table comprising three columns: time, activity/content and remarks. On the other hand all utterances concerning the textbook were transcribed literally. Furthermore, a focus was put on all utilizations of the textbook. Both, the use of the textbook by the students and by the teacher was taken into account. This is important for several reasons:

First of all, there is evidence from previous research that the teacher plays an integral part in mediating textbook use. Because of that, the teacher was included as a variable in the model of textbook use.

Secondly, the observation provides an insight into the way the teacher mediates textbook use in the classroom. It makes a difference if the students only use the textbook when they are told to by the teacher or if they use it of their own accord. This difference will become apparent through classroom observation.

Thirdly, the methodological triangulation provides a measure for the validity of the data. Collecting data on how the textbook has been used in the classroom makes it possible to compare the markings and comments of the students with the field notes. The degree of correspondence between these two sources relating to the use of the textbook in the classroom indicates how serious the students took their task.

While the method of highlighting and taking notes especially satisfies criterion 3 and at the same time aims at both, providing a precise record of the actual use of the textbook by students (criterion 1) as well as keeping biases low (criterion 2), the intention of the observation is threefold. On the one hand the idea is to lower biases that might be caused by the method of highlighting (criterion 2) and on the other the triangulation of two different data-sources provides a measure for the validity of the student's data.

In addition to the previously described methods interviews were conducted with selected students.

Data was collected for a period of three weeks in two 6th grade and two 12th grade classes in two German secondary schools. Within the German three partite school system, these schools are considered to be for high achieving students. All four classes were taught by different teachers.

The coding process followed the ideas of Grounded Theory by Strauss and Corbin (Strauss & Corbin, 1990). Categories were established in the process of analysing the data. Each highlighted section in the textbook was categorized according to the kind of block it belongs to (introductory tasks, exposition, worked example, kernels, exercises) (cf. Rezat, 2006), the activity it was involved in, and finally whether the use of the section was mediated by the teacher or not.

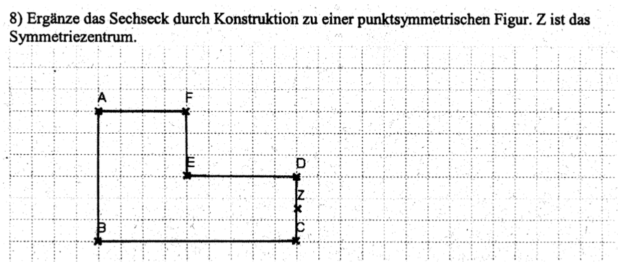
In order to understand the role of the mathematics textbook as an instrument within the activity system represented by the tetrahedron model Rabardel's (1995, 2002) theory of the instrument was used. As Monaghan (2007) points out, this theory has proven fruitful to provide insights into the use of new technologies as instruments for learning mathematics. According to Rabardel an instrument is a psychological entity that consists of an artefact component and a scheme component. In using the artefact with particular intentions the subject develops utilization schemes which are shaped by both, the artefact and the subject. Vergnaud (1998) suggests that schemes are characterized by two operational invariants: theorems-in-action and concepts-in-action. Since these two operational invariants are put forward in order to describe the representation of mathematical knowledge, it is not self-evident to apply them to knowledge related to the use of an artefact like the mathematics textbook. Therefore, it is suggested to generalize Vergnaud's notion of theorems-in-action and concepts-in-action to the notion of beliefs-in-action. As well as concepts-in-action beliefs are supposed to guide human behaviour by shaping what people perceive in any set of circumstances (Schoenfeld, 1998). Like theorems-in-action beliefs are propositions about the world that are thought to be true (Philipp, 2007). The appendix 'in-action' is supposed to underline that beliefs-in-action might be inferred from actions. They do not necessarily have to be expressed verbally. Because of its universality, the notion of beliefs-in-action offers an appropriate means to characterize operational invariants of utilization schemes linked to any artefact.

RESULTS

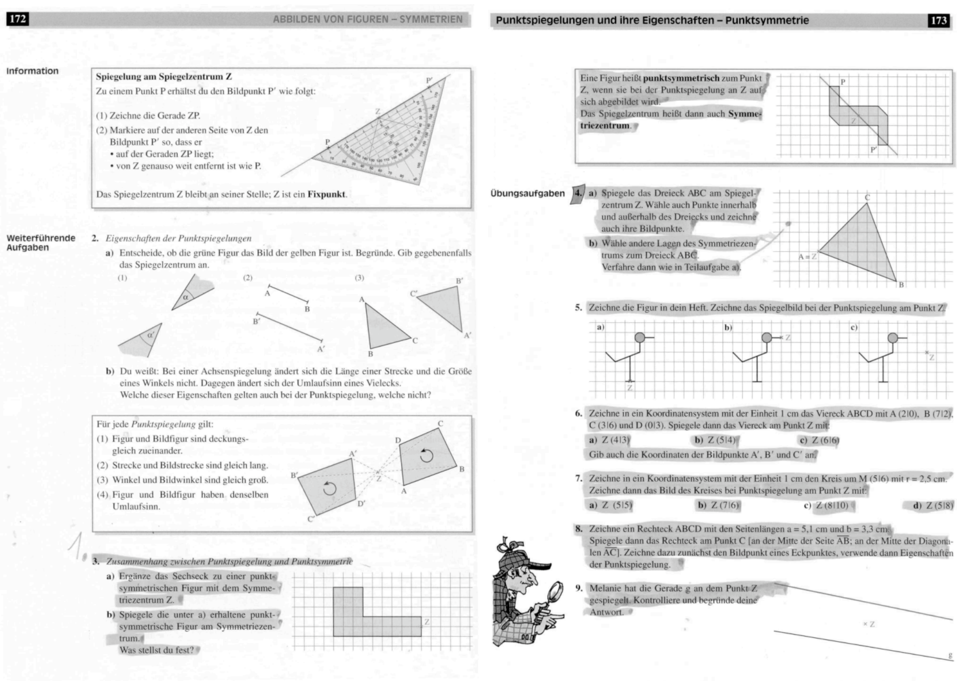
A first and a major result of the study is, that students do not only use the mathematics textbook when they are told to by the teacher. But, they also use the textbook self-directed. The following analysis focuses on utilizations of the mathematics textbook that students perform in addition to teacher mediated textbook use.

Students incorporate their mathematics textbook as an instrument into four activities: solving tasks and problems, consolidation, acquiring mathematical knowledge, and activities associated with interest in mathematics. From the data it was possible to reconstruct several individual utilization schemes of the mathematics textbook related to these activities. Comparing the individual schemes of different students related to the same activity revealed that some of the schemes were analogous in terms of the underlying beliefs-in-action. These schemes were generalized to utilization scheme types (UST). USTs are general in the way, that they allow to classify individual utilization schemes of the textbook into USTs and thus make individual utilizations comparable. Nevertheless, different students might show different USTs. The USTs are not general in the way that they are common to all students.

Solving tasks and problems is associated with activities where students utilize their mathematics textbook in order to get assistance with solving tasks and problems. Three different USTs were found related to this activity. It was observed that students repeatedly utilize specific blocks from the textbook as an assistance to solve tasks and problems. Worked examples and boxes with kernels were instrumentalized in most of the cases. This scheme could be traced back to the belief-in-action that a specific block from the textbook is useful in order to solve tasks and problems. It was also observed that students choose sections from the textbook that show similarities to the task. For example, Oliver is working on the following task that is not from the textbook:



He looks for assistance in the textbook and reads a task in the textbook that is located next to an image, which is identical to the image in the task. From this behaviour it can be inferred that Oliver expects information concerning the image next to it. In his case, the information is not useful for solving the task, because it is a task itself.


172 ABBILDEN VON FIGUREN – SYMMETRIEN

Information

Spiegelung am Spiegelzentrum Z
Zu einem Punkt P erhältst du den Bildpunkt P′ wie folgt:
(1) Zeichne die Gerade ZP.
(2) Markiere auf der anderen Seite von Z den Bildpunkt P′ so, dass er
• auf der Geraden ZP liegt;
• von Z genauso weit entfernt ist wie P.
Das Spiegelzentrum Z bleibt an seiner Stelle; Z ist ein **Fixpunkt**.

Weiterführende Aufgaben

2. *Eigenschaften der Punktspiegelungen*
a) Entscheide, ob die grüne Figur das Bild der gelben Figur ist. Begründe. Gib gegebenenfalls das Spiegelzentrum an.
(1) (2) (3)
b) Du weißt: Bei einer Achsenspiegelung ändert sich die Länge einer Strecke und die Größe eines Winkels nicht. Dagegen ändert sich der Umlaufsinn eines Vielecks.
Welche dieser Eigenschaften gelten auch bei der Punktspiegelung, welche nicht?

Für jede *Punktspiegelung* gilt:
(1) Figur und Bildfigur sind deckungsgleich zueinander.
(2) Strecke und Bildstrecke sind gleich lang.
(3) Winkel und Bildwinkel sind gleich groß.
(4) Figur und Bildfigur haben denselben Umlaufsinn.

3. *Zusammenhang zwischen Punktspiegelung und Punktsymmetrie*
a) Ergänze das Sechseck zu einer punktsymmetrischen Figur mit dem Symmetriezentrum Z.
b) Spiegele die unter a) erhaltene punktsymmetrische Figur am Symmetriezentrum.
Was stellst du fest?

Punktspiegelungen und ihre Eigenschaften – Punktsymmetrie 173

Eine Figur heißt **punktsymmetrisch** zum Punkt Z, wenn sie bei der Punktspiegelung an Z auf sich abgebildet wird.
Das Spiegelzentrum heißt dann auch **Symmetriezentrum**.

Übungsaufgaben

4. a) Spiegele das Dreieck ABC am Spiegelzentrum Z. Wähle auch Punkte innerhalb und außerhalb des Dreiecks und zeichne auch ihre Bildpunkte.
b) Wähle andere Lagen des Symmetriezentrums zum Dreieck ABC.
Verfahre dann wie in Teilaufgabe a).

5. Zeichne die Figur in dein Heft. Zeichne das Spiegelbild bei der Punktspiegelung am Punkt Z.
a) b) c)

6. Zeichne in ein Koordinatensystem mit der Einheit 1 cm das Viereck ABCD mit A (2|0), B (7|2), C (3|6) und D (0|3). Spiegele dann das Viereck am Punkt Z mit:
a) Z (4|3) b) Z (5|4) c) Z (6|6)
Gib auch die Koordinaten der Bildpunkte A′, B′ und C′ an.

7. Zeichne in ein Koordinatensystem mit der Einheit 1 cm den Kreis um M (5|6) mit r = 2,5 cm. Zeichne dann das Bild des Kreises bei Punktspiegelung am Punkt Z mit:
a) Z (5|5) b) Z (7|6) c) Z (8|10) d) Z (5|8)

8. Zeichne ein Rechteck ABCD mit den Seitenlängen a = 5,1 cm und b = 3,3 cm.
Spiegele dann das Rechteck am Punkt C [an der Mitte der Seite $\overline{AB}$; an der Mitte der Diagonalen $\overline{AC}$]. Zeichne dazu zunächst den Bildpunkt eines Eckpunktes, verwende dann Eigenschaften der Punktspiegelung.

9. Melanie hat die Gerade g an dem Punkt Z gespiegelt. Kontrolliere und begründe deine Antwort.

Fig. 2: Passage Oliver used from the textbook "because he was looking for something" (Griesel, Postel, & Suhr, 2003)

In order to get assistance with solving tasks and problems it was also observed that students search an adequate heading in the book and start reading from there until they find useful information. From this behaviour it was inferred that these students expect useful information related to a subject at the beginning of a lesson in the textbook.

All three USTs reveal that students are looking for information in the book that can be directly applied to the task. The only difference is the way they are approaching the information. Hardly ever does it seem like students want to understand the mathematics first and then apply it to the task.

Consolidation is associated with all activities that students perform in order to improve their mathematical abilities related to subject matters that were already dealt with in the mathematics class. One UST of students using their mathematics textbook for consolidation is strongly related to teacher mediated exercises from the textbook. They either recapitulate tasks and exercises from the book that the teacher mediated or they pick tasks and exercises that are adjacent to teacher-mediated exercises. This was traced back to the belief-in-action that effective practising means to do tasks and exercises that are similar to teacher-mediated exercises. If students pick tasks that are adjacent to teacher mediated tasks this is also supported by the belief-in-action that adjacent tasks in the textbook are similar. The use of specific blocks for consolidation was also observed. One UST is that students only read the boxes with the kernels of several lessons in the textbook.

So far, consolidation seems to comprise learning rules, recapitulating teacher mediated tasks and solving tasks that are similar to teacher mediated tasks respectively. But, it was also observed that students either utilize special parts at the end of a unit that are designed especially for recalling and practising the main issues of the unit or they scan the section in the book relating to the actual topic in the mathematics class and read different parts of it in order to consolidate their understanding of the topic. Both UST are less dependent on teacher mediation and show more proficiency in the utilization of the textbook.

Whereas consolidation related to previously treated topics, acquisition of knowledge is associated with activities where students use parts of the book that have not been a matter in the mathematics class so far. The UST identified in this context is that students use parts from the proximate lesson in the textbook. This is supported by the belief-in-action that the chronological succession of topics in the mathematics class will follow the order of the textbook.

Students also used parts of their textbook because they thought they were interesting. These utilizations are associated with activities related to interest in mathematics. In this case the UST is connected to the use of images and other salient elements from the book. Students either only look at the images or they read passages that are next to images or other salient elements. Looking just at the pictures does not seem to be associated with learning mathematics though. This UST usually is observed in the context of other utilizations of the textbooks. It seems like this UST is not based on a belief-in-action, but that salient elements in the textbook catch the attention of the students while there utilizing it for another purpose.

CONCLUSIONS

The activities the mathematics textbook is involved in do not only give an insight into student's utilizations of mathematics textbooks, but they also give an idea of what learning mathematics is about for students. The USTs show that learning mathematics with the mathematics textbook comprises activities as solving tasks and problems, consolidating mathematical knowledge and skills, acquiring new contents. The USTs show how the textbook is used as an instrument within these activities. Furthermore, these USTs reveal interesting insights into student's dispositions towards mathematics. Learning mathematics comprises mainly learning rules, applying rules and worked examples to tasks, and developing proficiency in tasks that are similar to teacher mediated tasks.

Consciousness about student's USTs could affect teacher's ways of implementing the mathematics textbook in the teaching process. Some USTs show that the use of mathematics textbooks by teachers in the classroom is an important reference for student's utilizations of the textbook. For example, the UST that is characterized by the utilization of tasks that are adjacent to teacher mediated tasks for consolidation is dependent on the mediation of tasks from the textbook by the teacher. Therefore, it is important that the teacher uses tasks from the textbook in order to support student's individual learning of mathematics. Another example is the anticipation of the next topic in the mathematics class by reading parts of the proximate lesson in the textbook. This UST shows that students belief that the course of the mathematics lessons will follow the order in the book. Accordingly, the textbook provides orientation for students, and it can therefore be considered important that teachers follow the succession of the topics in the book.

It was pointed out, that Valverde et al. (2002) argue that the structure of mathematics textbooks advances a distinct pedagogical model and is likely to have an impact on actual classroom instruction. From an ergonomical perspective it can be argued that the structure of the book also has an impact on the USTs of the students. This raises the question of how a textbook must be structured in order to promote desirable USTs.

Furthermore, this study provides evidence that Rabardels theory of the instrument is not only capable of conceptualizing human-computer-interaction, but is also applicable to non technological resources. The conceptualization of student-textbook-interaction on the basis of this theoretical framework provides interesting insights into different aspects of learning mathematics. The UST do not only provide a better understanding of student's utilizations of mathematics textbooks, but also reflect student's ways of learning mathematics. Furthermore, it can be inferred from student's USTs how the textbook is effectively used in the classroom by the teacher. Accordingly, a better understanding of student's utilizations of mathematics textbooks is a prerequisite for effective implementation of mathematics textbooks into teaching.

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AN INVESTIGATIVE LESSON WITH DYNAMIC GEOMETRY: A CASE STUDY OF KEY STRUCTURING FEATURES OF TECHNOLOGY INTEGRATION IN CLASSROOM PRACTICE

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Drawing on the research literatures concerning the classroom practice of mathematics teaching and technology integration in school mathematics, a previous CERME-5 paper (Ruthven, 2007) identified key structuring features – working environment, resource system, activity format, curriculum script, time economy – which shape patterns of technology integration into classroom practice and require teachers to develop their craft knowledge accordingly. In this paper, that conceptual framework is applied to an investigative lesson incorporating dynamic geometry use, employing evidence from classroom observations and teacher interviews. This analysis illuminates the many aspects of professional adaptation and development on which successful technology integration into classroom practice depends.

INTRODUCTION TO THE STUDY

From synthesis of relevant research literatures, a previous paper argued that successful integration of computer-based tools and resources into school mathematics depends on coordinating working environment, resource system, activity format and curriculum script to underpin classroom practice which is viable within the time economy (Ruthven, 2007). This paper will illustrate –and test– that conceptual framework by using it to analyse the practitioner thinking and professional learning surrounding a lesson incorporating the use of dynamic geometry.

The lesson was one of four cases investigated in a study of classroom practice incorporating dynamic geometry use (Ruthven, Hennessy & Deaney, 2008). In the original study, this specific case was followed up because the teacher concerned talked lucidly about his experience of teaching such a lesson for the first time, and displayed particular awareness of the potential of dynamic geometry for developing visuo-spatial and linguistic aspects of students' geometrical thinking.

This case has been chosen for further analysis because the teacher was unusually expansive in all his interviews, illuminating a range of aspects of practitioner thinking and professional learning. While an exhaustive case analysis in terms of the conceptual framework would require data to be collected with its use specifically in mind, the richness of the evidence from this case provides a convenient interim means of exploring its application to a concrete example.

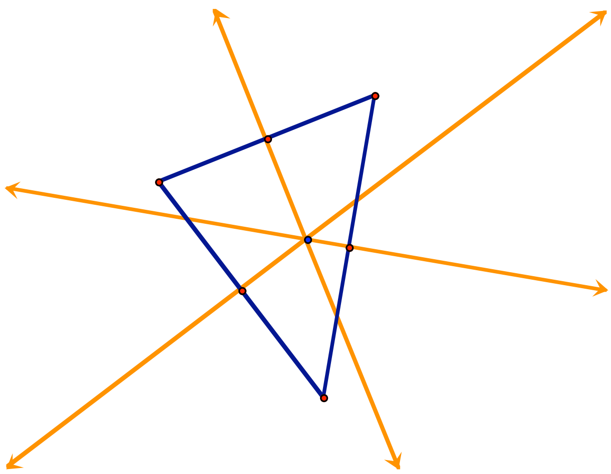
ORIENTATION TO THE LESSON

As the teacher explained when nominating the lesson, it had recently been developed in response to improved technology provision in the mathematics department prompting him to "*to explore some geometry*":

> *So we'd done some very rough work on constructions with compasses and bisecting triangles and then I extended that to Geometer's Sketchpad… on the interactive whiteboard using it in front of the class.*

He reported that the lesson (with a class in the early stages of secondary education) had started with him constructing a triangle, and then the perpendicular bisectors of its edges. The focus of the investigation which ensued had been on the idea that this construction might identify the 'centre' of a triangle:

> *And we drew a triangle and bisected the sides of a triangle and they noted that they all met at a point. And then I said: "Well let's have a look, is that the centre of a triangle?" And we moved it around and it wasn't the centre of the triangle, sometimes it was inside the triangle and sometimes outside.*



According to the teacher, one particularly successful aspect of the lesson had been the extent to which students actively participated in the investigation:

> *And they were all exploring; sometimes they were coming up and actually sort of playing with the board themselves… I was really pleased because lots of people were taking part and people wanted to come and have a go at the constructions.*

Indeed, because of the interest and engagement shown by students, the teacher had decided to extend the lesson into a second session, held in a computer room to allow students to work individually at a computer:

> *And it was clear they all wanted to have a go so we went into the computer room for the next lesson so they could just continue it individually on a computer… I was expecting them all to arrive in the computer room and say: "How do you do this? What do I have to do again?"… But virtually*

everyone… could get just straight down and do it. I was really surprised. And the constructions, remembering all the constructions as well.

For the teacher, then, this recall by students of ideas from the earlier session was another aspect of the lesson's success. In terms of the specific contribution of dynamic geometry to this success, the teacher noted how the software supported exploration of different cases, and overcome the practical difficulties which students encountered in using classical tools to attempt such an investigation by hand:

You can move it around and see that it's always the case and not just that one off example. But I also think they get bogged down with the technicalities of drawing the things and getting their compasses right, and [with] their pencils broken.

But the teacher saw the contribution of the software as going beyond ease and accuracy; using it required properties to be formulated precisely in geometrical terms:

And it's the precision of realising that the compass construction… is about the definition of what the perpendicular bisector is… And Geometer's Sketchpad forces you to use the geometry and know the actual properties that you can explore.

These, then, were the terms in which the original lesson was nominated as an example of successful practice. This nomination was followed up by studying a later lesson along similar lines through classroom observations and teacher interviews. The observed lesson was conducted over two 45-minute sessions on consecutive days with a Year 7 class of students (aged 11-12) in their first year of secondary education.

WORKING ENVIRONMENT

The use of ICT in teaching often involves changes in the working environment of lessons: change of room location and physical layout, change in class organisation and classroom procedures.

Each session of the observed lesson started in the normal classroom and then moved to a nearby computer suite, a modification of the pattern originally reported. This movement between rooms allowed the teacher to follow a particular activity cycle common to each session, shifting working environment to match changing activity format. The classroom was equipped with a single computer linked to a ceiling-mounted projector directed towards a whiteboard at the front: this supported use of computer-based resources within whole-class activity formats. However, only in the computer suite was it possible for students to work individually at a machine.

Even though the suite was also equipped with a projectable computer, starting sessions in the teacher's own classroom was expedient for several reasons. Doing so avoided disruption to the established routines underpinning the smooth launch of lessons. Moreover, the classroom provided an environment more conducive to sustaining effective communication during whole-class activity and to maintaining the attention of students. Whereas in the computer suite each student was seated behind a sizeable monitor perched on a desktop computer unit, so blocking lines of sight and placing diversion at students' fingertips, in the classroom the teacher could introduce the lesson *"without the distraction of computers in front of each of them"*.

It was only recently that the classroom had been refurbished and equipped, and a neighbouring computer suite established for the exclusive use of the mathematics department. The teacher contrasted this new arrangement favourably in terms of the easier and more regular access to technology that it afforded, and the consequent increase in the fluency of students' use:

> *Before… you'd book a computer suite, you'd go in and then… you'[d] just not get anywhere, because the whole lesson's been sorting out logging on, sorting out how to use [the software]… And [now] having the access to it so easily and readily just makes a huge difference.*

New routines were being introduced to students for opening a workstation, including logging on to the school network, using shortcuts to access resources, and maximising the document window. Likewise, routines were being developed for closing sessions in the computer suite. Towards the end of each session, the teacher prompted students to plan to save their files and print out their work, advising them that he'd *"rather have a small amount that you understand well than loads and loads of pages printed out that you haven't even read"*. He asked students to avoid rushing to print their work at the end of the lesson, and explained how they could adjust their output to try to fit it onto a single page; he reminded them to give their file a name that indicated its contents, and to put their name on their document to make it easy to identify amongst all the output from the single shared printer.

RESOURCE SYSTEM

New technologies have broadened the types of resource available to support school mathematics. Nevertheless, there is a great difference between a collection of resources and a coherent system.

The department maintained its own schemes of work under continuous development, with teachers encouraged to explore new possibilities and report to colleagues. This meant that they were accustomed to integrating material from different sources into a common scheme. However, so wide was the range of computer-based resources currently being trialled that our informant (who was head of department) expressed concern about incorporating them effectively into departmental schemes:

> *At the moment we're just dabbling in [a variety of technologies and resources] when people feel like it, but we're moving towards integrating [them] into schemes of work now… I'm slightly worried that we've got so much… It's getting everybody familiar with it all.*

In terms of coordinating use of old and new technologies, work with dynamic geometry was seen as complementing established work on construction by hand, by strengthening attention to the related geometric properties:

> *I thought of Geometer's Sketchpad [because] I wanted to balance the being able to actually draw [a figure] with pencil and compasses and straight edges, with also seeing the geometrical facts about it as well. And sometimes [students] don't draw it accurately enough to get things like that all the [perpendicular bisectors] meet at the orthocentre[1] of the circle.*

The accuracy, speed and manipulative ease of dynamic geometry facilitated geometrical investigations which were difficult to undertake by hand:

> *[It] takes hours and hours if you try and do that by pencil and paper… So just that power of Geometer's Sketchpad to move the triangle around and try different triangles within seconds was fantastic. Ideal for this sort of exploration.*

Nevertheless, the teacher felt that old and new tools lacked congruence, because certain manual techniques appeared to lack computer counterparts. Accordingly, old and new were seen as involving different methods and having distinct functions:

> *When you do compasses, you use circles and arcs, and you keep your compasses the same. And I say to them: "Never move your compasses once you've started drawing."… Well Geometer's Sketchpad doesn't use that notion at all… So it's a different method.… I don't think there's a great deal of connection. I don't think it's a way of teaching constructions, it's a way of exploring the geometry.*

Equally, some features of computer tools were not wholly welcome: students could be deflected from the mathematical focus of a task by overconcern with presentation. During this lesson the teacher had tried out a new technique for managing this, by briefly projecting a prepared example to show students the kind of document that they were expected to produce, and illustrating appropriate use of colour coding:

> *They spend about three quarters of the lesson making the font look nice and making it all look pretty [but] getting away from the maths.… I've never tried it before, but that showing at the end roughly what I wanted them to have would help. Because it showed that I did want them to think about the presentation, I did want them to slightly adjust the font and change the colours a little bit, to emphasise the maths, not to make it just look pretty.*

Here we see the development of sociomathematical norms for using new technologies, and classroom strategies for establishing and maintaining these norms. Likewise, the way in which dynamic geometry required clear instructions to be given in precise mathematical terms was conveyed as being its key characteristic:

I always introduce Geometer's Sketchpad by saying: "It's very specific, you've got to tell it. It's not just drawing, it's drawing using mathematical rules."… They're quite happy with that notion of… the computer only following certain clear instructions.

ACTIVITY FORMAT

Classroom activity is organised around formats for action and interaction which frame the contributions of teacher and students to particular lesson segments (Burns & Anderson, 1987). The crafting of lessons around familiar activity formats and their supporting classroom routines helps to make them flow smoothly in a focused, predictable and fluid way (Leinhardt, Weidman & Hammond, 1987). This leads to the creation of prototypical activity structures or cycles for particular styles of lesson.

Each session of the observed lesson followed a similar activity cycle, starting with teacher-led activity in the normal classroom, followed by student activity at individual computers in the nearby computer suite, and with change of rooms during sessions serving to match working environment to activity format. Indeed, when the teacher had first nominated this lesson, he had remarked on how it combined a range of classroom activity formats to create a promising lesson structure:

There was a bit of whole class, a bit of individual work and some exploration, so it's a model that I'd like to pursue because it was the first time I'd done something that involved quite all those different aspects.

In discussing the observed lesson, however, the teacher highlighted one aspect of the model which had not functioned as well as he would have liked: the fostering of discussion during individual student work. He identified a need for further consideration of the balance between opportunities for individual exploration and productive discussion, through exploring having students work in pairs:

There was not as much discussion as I would have liked. I'm not sure really how combine working with computers with discussing. You can put two or three [students] on a computer, which is what you might have done in the days when we didn't have enough computers, but that takes away the opportunity for everybody to explore things for themselves. Perhaps in other lessons… as I develop the use of the computer room I might decide… [to] work in pairs. That's something I'll have to explore.

At the same time, the teacher noted a number of ways in which the computer environment helped to support his own interactions with students within an activity format of individual working. Such opportunities arose from helping students to identify and resolve bugs in their dynamic geometry constructions:

[Named student] had a mid point of one line selected and the line of another, so he had a perpendicular line to another, and he didn't actually notice which is worrying… And that's what I was trying to do when I was

going round to individuals. They were saying: "Oh, something's wrong." So I was: "Which line is perpendicular to that one?"

Equally, the teacher was developing ideas about the pedagogical affordances of text-boxes, realising that they created conditions under which students might be more willing to consider revising their written comments:

And also the fact that they had a text box… and they could change it and edit it. They could actually then think about what they were writing, how they describe, I could have those discussions. With handwritten, if someone writes a whole sentence next to a neat diagram, and you say: "Well actually, what about that word? Can you add this in?" You've just ruined their work. But with technology you can just change it, highlight it and add on an extra bit, and they don't mind.

This was helping him to achieve his goal of developing students' capacity to express themselves clearly in geometrical terms:

I was focusing on getting them to write a rule clearly. I mean there were a lot writing "They all meet" or even, someone said "They all have a centre."… So we were trying to discuss what "all" meant, and a girl at the back had "The perpendicular bisectors meet", but I think she'd heard me say that to someone else, and changed it herself. "Meet at a point": having that sort of sentence there.

CURRICULUM SCRIPT

In planning and conducting lessons on a topic, teachers draw on a loosely ordered model of relevant goals and actions that guides their teaching. This forms what has been termed a 'curriculum script' – where 'script' is used in the psychological sense of a form of event-structured cognitive organisation, which includes variant expectancies of a situation and alternative courses of action (Leinhardt, Putnam, Stein & Baxter, 1991). This script includes tasks to be undertaken, representations to be employed, activity formats to be used, and student difficulties to be anticipated.

The observed lesson followed on from earlier ones in which the class had undertaken simple constructions with classical tools: in particular, using compasses to construct the perpendicular bisector of a line segment. Further evidence that the teacher's script for this topic originated prior to the availability of dynamic geometry was his reference to the practical difficulties which students encountered in working by hand to accurately construct the perpendicular bisectors of a triangle. His evolving script now included knowledge of how software operation might likewise derail students' attempts to construct perpendicular bisectors, and of how such difficulties might be turned to advantage in reinforcing the mathematical focus of the task:

Understanding the idea of perpendicular bisector… you select the line and the [mid]point… There's a few people that missed that and drew random *lines… And I think they just misunderstood, because one of the awkward things about it is the selection tool. If you select on something and then you select another thing, it adds to the selection, which is quite unusual for any Windows package… So you have to click away and de-select things, and that caused a bit of confusion, even though I had told them a lot. But… quite a few discussions I had with them emphasised which line is perpendicular to that edge… So sometimes the mistakes actually helped.*

Equally, the teacher's curriculum script anticipated that students might not appreciate the geometrical significance of the concurrence of perpendicular bisectors, and incorporated strategies for addressing this:

They didn't spot that [the perpendicular bisectors] all met at a point as easily… I don't think anybody got that without some sort of prompting. It's not that they didn't notice it, but they didn't see it as a significant thing to look for… even though there were a few hints in the worksheet that that's what they were supposed to be looking at, because I thought that they might not spot it. So I was quite surprised… that they didn't seem to think that three lines all meeting at a point was particularly exceptional circumstances. I tried to get them to see that… three random lines, what was the chance of them all meeting at a point.

The line of argument alluded to here was one already applicable in a pencil and paper environment. Later in the interview, however, the teacher made reference to another strategy which brought the distinctive affordances of dragging the dynamic figure to bear on this issue:

When I talked about meeting at a point, they were able to move it around, and I think there's more potential to do that on the screen.

Likewise, his extended curriculum script depended on exploiting the distinctive affordance of the dynamic tool to explore how dragging the triangle affected the position of the 'centre'.

This suggests that the teacher's curriculum script was evolving through experience of teaching the lesson with dynamic geometry, incorporating new mathematical knowledge specifically linked to mediation by the software. Indeed, he drew attention to a striking example of this which had arisen from his question to the class about the position of the 'centre' when the triangle was dragged to become right angled:

Teacher: What's happening to the [centre] point as I drag towards 90 degrees? What do you think is going to happen to the point when it's at 90?

Student: The centre's going to be on the same point as the midpoint of the line.

Teacher [with surprise]: Does it always have to be at the midpoint?

[Dragging the figure] Yes, it is! Look at that! It's always going to be on the midpoint of that side.… Brilliant!

Reviewing the lesson, the teacher commented on this episode, linking it to distinctive features of the mediation of the task by the dynamic figure:

I don't know why it hadn't occurred to me, but it wasn't something I'd focused on in terms of the learning idea, but the point would actually be on the mid point.… As soon as I'd said it I thought "Of course!" But you know, in maths there's things that you just don't really notice because you're not focusing on them. And… I was just expecting them to say it was on the line. Because when you've got a compass point, you don't actually see the point, it's just a little hole in the paper… But because the point is actually there and quite clear, a big red blob, then I saw it was exactly on that centre point, and that was good when they came up with that.

In effect, his available curriculum script did not attune the teacher to this property. One can reasonably hazard that this changed as a direct result of this episode.

TIME ECONOMY

Assude (2005) examines how teachers seek to improve the 'rate' at which the physical time available for classroom activity is converted into a didactic time measured in terms of advance of knowledge. The adaptation and coordination of working environment, resource system, activity format and curriculum script are very important in improving this didactic 'return' on time 'investment'.

In respect of this time economy, a basic consideration of physical time for the teacher in this study was the proximity of the new computer suite to his normal classroom:

I'm particularly lucky being next door… If I was upstairs or something like that, it would be much harder; it would take five minutes to move down.

However, a more fundamental feature of this case was the degree to which the teacher measured didactic time in terms of progression towards securing student learning rather than pace in covering a curriculum. At the end of the first session, he linked his management of time to what he considered to be key learning processes:

It's really important that we do have that discussion next lesson. Because they've seen it. Whether they've learned it yet, I don't know… They're probably vaguely aware of different properties and they've explored it, so it now needs to be brought out through a discussion, and then they can go and focus on writing things for themselves. So the process of exploring something, then discussing it in a quite focused way, as a group, and then writing it up… They've got to actually write down what they think they've learned. Because at the moment, I suspect… they've got vague notions of what they've learned but nothing concrete in their heads.

A further crucial consideration within the time economy is instrumental investment. The larger study from which this case has been derived showed that the ways in which teachers incorporated dynamic geometry into classroom activity were influenced by their assessments of costs and benefits. Essentially, teachers were willing to invest time in developing students' instrumental knowledge of dynamic geometry to the extent that they saw this as promoting students' mathematical learning. As already noted, this teacher saw working with the software as engaging students in disciplined interaction with a geometric system. Consequently, he was willing to spend time to make them aware of the construction process underlying the dynamic figures used in lessons:

> *I very rarely use Geometer's Sketchpad from anything other than a blank page. Even when I'm doing something in demonstration… I always like to start with a blank page and actually put it together in front of the students so they can see where it's coming from.*

Equally, this perspective underpinned his willingness to invest time in familiarising students with the software, capitalising on earlier investment in using classical tools:

> *That getting them used to the program beforehand, giving a lesson where the aim wasn't to do that particular maths, but just for them to get familiar with it… was very helpful. And also they're doing the constructions by hand first, to see, getting all the words, the key words, out of the way.*

As this recognition of a productive interaction between learning to use old and new technologies indicates, this teacher also took an integrative perspective on the 'double instrumentation' entailed. Indeed, this was demonstrated earlier in his concern with the complementarity of old and new as components of a coherent resource system.

CONCLUSION

This analysis of a lesson incorporating dynamic geometry illuminates the influence of the key structuring features of working environment, resource system, activity format, curriculum script and time economy on technology use. Although only employing a dataset conveniently available from earlier research, it starts to show the complex character of the professional adaptation on which technology integration into the classroom practice of school mathematics depends. This points to the value of conducting further studies in which data collection (as well as analysis) is guided by the conceptual framework developed in this paper and its predecessor.

NOTES

[1] The point at which the perpendicular bisectors of the sides of a triangle meet is the 'circumcentre' in English. However, in the course of the interview, the teacher referred to this centre as the 'orthocentre'. Note that it is now many years since reference to these (and other) terms – which distinguish the different 'centres' of a triangle – was removed from the school mathematics curriculum in England.

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RELATIONSHIP BETWEEN DESIGN AND USAGE OF EDUCATIONAL SOFTWARE: THE CASE OF APLUSIX

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In this contribution, we are interested in the design process of Aplusix, a microworld for the learning of algebra and in the impact of usages on this process. In the first part, we present general principles that seem to be guiding the overall design process of the system and the development of tree representation of algebraic expressions, which has been added recently. The second part is devoted to a design and implementation of a learning scenario involving Aplusix. Examples of impact of this empirical study on the software design choices are discussed.

Key words: *Aplusix, algebra, tree representation, pedagogical scenario*

INTRODUCTION

The research reported in this paper is carried out in the framework of the ReMath project (http://remath.cti.gr) addressing the issue of using technologies in mathematics classes "*taking a 'learning through representing' approach and focusing on the didactical functionality of digital media*". The digital media at the core of this research is Aplusix, software designed to help students learn algebra. The work has been developed in three phases:

(1) *Design and implementation of a new representation of algebraic expressions*. During this phase, fundamental choices for a representation of expressions in a form of a tree were made collaboratively through interactions between computer scientists and didacticians of mathematics: on the one hand, computer scientists make sure that the new developments comply with general principles of the software, on the other hand, didacticians ensure that these choices are based on didactical and epistemological hypotheses. The choice of theoretical frameworks in both domains has an impact on functionalities of the tree representation. This design phase is presented in the following section.

(2) *Design of a pedagogical scenario*. Based on the choices made in the design phase, didacticians designed a pedagogical scenario to explore possible contributions of this new representation to the learning of algebra. The scenario has to take account of institutional constraints in order to implement it in ordinary classes. The design of scenario may lead to reconsidering certain choices concerning the new representation, or suggesting other. Such cases will be presented further in the paper.

(3) *Experimentation*. The scenario has been experimented in three different classes, which allowed validating underlying didactical hypotheses, as well as assessing the

way students manipulate this new representation. This phase is discussed in the last part of the paper.

DESIGN AND DEVELOPMENT OF APLUSIX

When developing computer-based learning environments, designers need to make choices at the interface level and thus at the level of the internal universe of the environment. Thus pieces of knowledge implemented in such an environment will live not only under constraints of the didactical transposition (Chevallard 1985), but also under other constraints proper to the environment resulting from what Balacheff (1994) calls *computational transposition*. Thus, designers of computer-based learning environments have to respond to at least two types of requirements. First, they need to respect basic principles that are characteristic of the environment. The second type is related to the practice of the piece of knowledge in the institution in which it will be used.

Principles governing a design of software are not always made explicit and choices made are rarely explicitly linked to these principles. In what follows, we present a study carried out in an attempt to make explicit principles and choices that were guiding designers of Aplusix (aplusix.imag.fr), software for learning algebra, when they were developing tree representation of algebraic expressions.

GENERAL DESIGN PRINCIPLES OF APLUSIX

Aplusix software (Nicaud et al., 2003, 2004) has been developed since 1980s. A new mode of representation of algebraic expressions, a tree representation, is being added to this software. As was already mentioned above, the new developments must not affect the coherence of the whole software and thus have to comply with fundamental principles that guide the design and development of Aplusix. Three main design principles have been identified:

(1) *The student is free to write algebraic expressions*. This principle, influenced by research in the domain of interactive learning environments, considering mainly microworlds, resulted in the development of an editor of algebraic expressions and in the necessity to consider and deal with students' errors.

However, freedom in manipulating algebraic expressions is limited by constraining the selection of sub-expressions, based on the syntactic and semantic dimensions of expressions, which seems to be another important design principle and that can be formulated as follows:

(2) *In manipulating algebraic expressions, their syntactic and semantic dimensions are taken into account*. For example, given the expression 2+3x, it is not possible to select 2+3 as a sub-expression. This principle brings the idea of scaffolding since this choice aims at helping understand algebraic expressions and make their manipulation easier.

As regards the interaction between a student and a system, there are two modes of interaction: (1) a test mode in which the student does not get any feedback from the system, and (2) a training mode, in which a feedback is provided both in terms of equivalence of a student's expression and the given one, and in terms of the correct end of the exercise. Thus the third principle is:

(3) *In a training mode, scaffolding should be provided by the system.* Scaffolding in the training mode requires taking decisions about validation of student's answers. It is important to clarify at this point that Aplusix recognizes 4 basic types of exercises: calculate, expand and simplify, factor and solve (equation, inequality or system of equations or inequalities). For these types of exercises, these decisions have been implemented. For example, for the "solve equation" exercise, it has been decided that the expression $x = 2/4$ will not be accepted as it is written in a non-simplified form, but will not be rejected either as it is not incorrect. Therefore a feedback message is sent to the student saying that the equation is almost solved.

DESIGN AND DEVELOPMENT OF TREE REPRESENTATION IN APLUSIX

The decision to implement a new representation system into the existing Aplusix software was taken in relation with the ReMath project focusing on representations of mathematical concepts in educational software. Two possibilities were considered: tree and graphical representations. The reasons for choosing the development of tree representation system are numerous (Bouhineau et al. 2007): (1) from an epistemological point of view, trees are natural representations of algebraic expressions; (2) from a didactical point of view, the introduction of a new register of representation would allow creating activities requiring an interplay between registers, which would enhance learning of algebraic expressions (Duval 1993); (3) from a point of view of computer science, trees are fundamental objects used to define data structures. Indeed, internal objects used in Aplusix to represent algebraic expressions and their visual properties are trees; (4) graphical representation of algebraic expressions is available in a few educational systems, while tree representation is scarcer.

Let us note first that the fundamental choices related to the tree representation were discussed during several meetings among developers (computer scientists and engineers) and didacticians.

Different modes of tree representation

The first idea was to develop the tree representation in a way that the student can see the articulation between the usual representation of an expression and a tree representing it: given an expression in a usual representation, a tree representation is provided progressively by the system, according to the student's command. A "mixed representation" mode has thus been designed where each leaf of a tree is a usual representation of an expression that can be expanded in a tree by clicking at the "+"

button that appears when the mouse cursor is near a node; a tree, or a part of a tree, can be collapsed into a usual representation by clicking at the "-" button that appears when the mouse cursor is near a node. The developers considered this idea interesting from the learning point of view. However, it was in contradiction with the principle 1, according to which it was necessary to let the student edit freely a tree. The development of a "free tree representation" mode, where the student can freely built trees, brought new difficulties the developers had to face: notion of erroneous operator, representation of parentheses, difficulties related to the "minus" sign, to the square root… These difficulties and the ways the developers have coped with them are described elsewhere (Trgalova and Chaachoua 2008).

Based on the principle 3, the developers wished to implement an editing mode providing scaffolding to the student. Design and implementation of scaffolding requires to define new kinds of exercises that would be recognized by the system and the means of validation of these exercises. We will discuss some of these choices below. It led also to the implementation of a "controlled tree representation" mode with constraints and scaffolding when a tree is edited: internal nodes must be operators and leaves must be numbers or variables. The arity of operators must be correct. In the current prototype of Aplusix, 3 modes of editing trees are thus available: free, controlled and mixed representations.

Choices of criteria for validating a student's answer

According to the principle 3, when the student builds a tree in the free tree representation mode, the system should provide her/him with a feedback. Decisions about the conditions for a tree to be accepted as correct had to be taken and implemented. The student's tree is compared with the expected one: (1) when, after normalisation of the minus signs (transformation of all minus signs in opposite), the trees are identical, then the student tree is accepted; (2) when the two trees differ only by commutation, the student's tree is not accepted, but a specific message indicates that there is a problem with order; (3) when there is neither identity between the trees (case 1) nor commutation (case 2) but the two trees represent equivalent expressions, a message is generated indicating that the student's tree is equivalent but not the expected one; (4) when there is no equivalence between expressions represented by the trees, another message is generated indicating that the answer is not correct.

These choices were made by one of the developers based on *fundamental issues* present in Aplusix such as *the notion of equivalence, the notion of commutation* and *of associativity*. They are considered as *a first stage choices* that can be discussed and analysed from the didactical point of view, both in terms of messages to be generated and of considering different cases of behaviour.

PEDAGOGICAL SCENARIO

Before presenting a pedagogical scenario we designed in order to validate design choices for the tree representation of expressions in Aplusix, we discuss some theoretical considerations that underpin the scenario.

According to Sfard (1991), mathematical notions can be conceived in two different ways: structurally as objects, and operationally as processes. An object conception of a notion focuses on its form while a process conception focuses on the dynamics of the notion. Algebraic expression, when conceived operationally, refers to a computational process. For example, the expression 5x-2 denotes a computational process "multiply a number by 5, and then subtract 2", which can be applied to numerical values. When an expression is conceived structurally, it refers to a set of objects on which operations can be performed. For example, 5x-2 denotes the result of the computational process applied to a number x. It also denotes a function that assigns the value 5x-2 to a variable x. Yet, in the French high school, the operational conception of algebraic expressions prevails in the teaching of algebra. Specific activities are needed to favour the distinction between these two conceptions of an algebraic expression. Examples of such activities are describing the expression in natural language, which requires considering the structure of the expression, or using tree representation of an expression, which highlights its form.

Semiotic representation is of major importance in any mathematical activity since mathematical concepts are accessible only by means of their representations. Duval (1995) calls "*register of representation*" any semiotic system allowing to perform three cognitive activities inherent to any representation: formation, treatment and conversion. These activities correspond to different cognitive processes and cause numerous difficulties in learning mathematics. Duval (2006) claims that while treatment tasks are more important from the mathematical point of view, conversion tasks are critical for the learning. Consequently, conceptualisation of mathematical notions requires manipulating of several registers for the same notion allowing to distinguish between a notion and its representations. As Duval (1993) says, the conceptualisation relies upon the articulation of at least two registers of representation, and this articulation manifests itself by rapidity and spontaneity of the cognitive activity of conversion between registers. Yet, school mathematics gives priority to teaching rules concerning both formation of semiotic representations and their treatment. The amount of activities of conversion between registers is negligible, although they represent cognitive activities that are the most difficult to grasp by students.

Motivated by these considerations, in the design of our pedagogical scenario, we decided to take into account three semiotic registers of representation of algebraic expressions: natural language register (NLR), usual register (UR) and tree register (TR) and to design activities of formation, treatment and conversion between these registers. The pedagogical scenario thus aims at helping the students grasp the structure of algebraic expressions by means of introducing TR and articulating it with

UR and NLR. The following hypothesis underpins the scenario: the introduction of TR and its articulation with NLR and UR will have a positive impact on students' mastering of the usual register of representation of algebraic expressions, which is the one taught in school algebra. The scenario is composed from 4 units: *pre-test*, *learning*, *assessing*, and *post-test* (cf. Table 1). The *pre-test* aimed at diagnosing students' difficulties in algebra, especially those related to the structural aspect of expressions. On the other hand, the results of the pre-test compared to those of the post-test should provide us with evidence about the efficiency of the pedagogical scenario. Two kinds of activities are proposed in the pre-test: (1) classical school algebra exercises (calculate, expand and simplify, factor), which are, in Duval's terms, treatment tasks in the register of usual representation, and (2) communication games between students proposing, in Duval's terms, activities of conversion between UR and NLR. The aim of the *learning* unit is to introduce the students to TR, a new register of representation of expressions, as well as to articulate it with the already familiar registers, namely NLR and UR. Then, conversion activities between TR and NLR and UR respectively are proposed. Most of the activities are to be done in a computer lab with Aplusix in the training mode. Eventually, simple tasks of treatment in TR are proposed to assess the mastery of the new register of representation by students. The unit called *assessing* aims at evaluating to what extent TR and conversion tasks between the registers are mastered by the students after having done activities of the *learning* unit. The evaluation is organized in the form of communication games between students similar to those from the pre-test, but this time, TR is involved in the tasks. In the *post-test*, tasks similar to those from the pre-test are proposed in order to enable a comparison of results. Confronting results obtained at the two tests should provide us with evidence confirming or not the underlying hypothesis.

	Activities	*Description*	*Environment*	*Duration*
Pre-test	Treatment in UR	Calculate, Factor Expand and simplify	Aplusix	50 min
	Conversion NLR ↔ UR	Communication games	Paper & pencil	30 min
Learning	Introduction to TR	Scenario TR introduction	Aplusix in video projection	55 min
	Conversion NLR ↔ TR	Conversion NLR → TR Conversion TR → NLR	Aplusix: controlled then free mode Paper & pencil	90 min
	Conversion UR ↔ TR	Conversion UR → TR Conversion TR → UR	Aplusix: controlled then free mode	80 min
	Treatment in TR	Calculate in TR	Aplusix with second	20 min

		Simplify in TR	view	
Ass.	Formation TR Conversion TR ↔ NLR (UR)	Communication games	Aplusix: free mode Paper & pencil	55 min
Post-test	Treatment in RU	Calculate, Factor Expand and simplify	Aplusix	30 min
	Conversion NLR ↔ UR	Communication games	Paper & pencil	20 min

Table 1. Structure of the pedagogical scenario.

EXPERIMENTATION

The scenario was proposed to 3 teachers with a possibility to adapt it to the constraints of their class. In this section, we present one of the experiments that took place in a Grade 10 class (15 years old students) in November 2007.

The pre-test revealed expected errors in treatment tasks within UR, in particular errors showing difficulties to take account of the structure of algebraic expressions, e.g., transforming 2+3x in 5x, and errors with handling powers and minus sign, e.g., transforming $3(-5)^2$ in -3×5^2 or in $\pm3^2\times5^2$. On the other hand, we were surprised by the results obtained in communication games. Algebraic expressions given in UR were described in NLR by the students, but with characteristics of an oral register, i.e., the students described actions allowing to obtain the initial expression (cf. Table 2). This register is based on language structure used to "read" an expression in UR. It presents two specificities: left-to-right reading and presence of implicit elements.

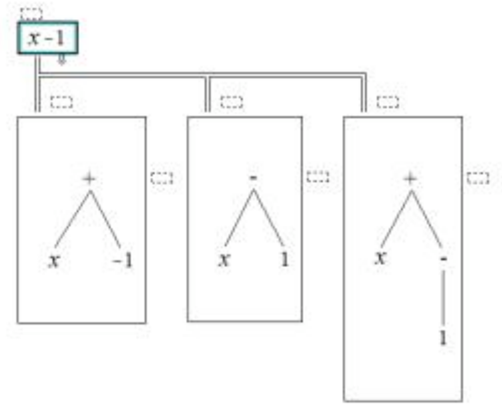
Expression given in UR	***Student emitting a message***		***Student receiving a message***	
	Register	*Examples of messages*	*Correct in UR*	*Wrong in RU*
2x – y	Oral (left-to-right)	"2 x minus y"	14	0
$2x - y^2$	Oral with ambiguity	"2 x minus y squared"	22	4
$\frac{(3x+2)(3x-1)}{a-(x+2)}$	Oral with brackets explicitly stated	"open a bracket, 3 x plus 2, close the bracket, open a bracket, 3 x minus 1, close the bracket, all this over a minus, open a bracket, x plus 2, close the bracket"	7	1

	Oral with brackets explicitly stated and with ambiguity	“open a bracket, 3 x plus 2, close the bracket, open a bracket, 3 x minus 1, close the bracket, over a minus, open a bracket, x plus 2, close the bracket”	19	1
Total			62	6

Table 2. Conversion from UR into NLR.

All messages result from the oral register and they accentuate operational aspect of the expressions rather than structural one. Moreover, more than 66% of messages are ambiguous. Despite of the ambiguities, most of pairs succeeded the game thanks to implicit codes of the oral register the students share and understand and which result from didactical contract (Brousseau 1997). Thus, the goal we assigned to the communication games, namely to lead students to become aware of the limits of the oral register they use in algebra, which does not take into account the structural aspect of expressions, was not achieved.

The learning unit started by an introductory session aiming at introducing tree representation to the students. The teacher asked one of the designers of the pedagogical scenario to manage this session since he did not feel comfortable enough with the new representation implemented in the software although he uses Aplusix on a regular basis with his students. This introductory session allowed discussing with the students specificities of the tree representation of expressions and introducing vocabulary related to this new register (branch, leave, operator, argument…). Particular attention was paid to reading the expressions. Thus for example, the expression x+2y was read as “the sum of x and of the product of 2 by y”, which accentuates the structure of the expression, instead of “x plus 2 y” highlighting its operational aspect. A particularity of the tree register residing in the fact that several different trees can represent a same algebraic expression was also discussed with the students based on the following example showing different meanings of “minus” sign (Fig. 1):



In the expression x-1, the minus sign can be conceived in three different ways leading to three different trees (this difference is hardly visible in UR):

- Sign of a negative number (tree on the left);
- Binary operator "difference" (tree in the middle);
- Unary operator "opposite number" (tree on the right).

Figure 1. Three different meanings of minus sign.

The rest of the scenario was shortened in order for the teacher to be in line with the global pedagogical program shared by all Grade 10 classes in the school. The teacher decided to individualize the implementation of the scenario according to the students in the following way: conversion NLR□TR and UR□TR in controlled mode only (only one group, denoted G1); conversion TR□NLR assigned as homework (whole class); treatment in TR optional (a few students with severe difficulties in algebra).

The G1 group was formed from rather low attaining students. The results obtained in the conversion tasks TR□NLR showed a significant difference between the two groups (cf. Table 3). These results can be considered as evidence proving efficiency of the work on conversion tasks NLR/UR□TR.

	Answer in NLR with structural aspect	*Answer in NLR with operational aspect*
G1 15 students having worked on conversion tasks with Aplusix in controlled mode	10	5
G2 15 students who have not benefited from the work on conversion tasks	3	12

Table 3. Students' answers to the conversion tasks TR→NLR.

As we mentioned above, the scenario, and thus the new prototype of Aplusix, had been tested in three classes. Feedbacks from students and teachers led the developers to re-examine some choices, which allowed some adaptations and improvements at the interface of Aplusix. Let us take the example of the "second view" functionality that enables visualizing a given algebraic expression represented in two registers at the same time. Initially, the second view displayed only a current step of the transformation. Observing the students using this functionality, we realized that when a student performs the next transformation step, the representation in the second view is updated and the student cannot observe the effects of the transformation in the second register. For this reason, the developers were asked to redesign this functionality in a way for the student to be able to observe the transformation s/he has performed in both registers. At present, the second view displays both current and previous steps.

CONCLUSION

The example of the design and implementation of tree representation of algebraic expressions presented in this contribution shows that the decision to introduce a new register of representation has been motivated by the didactical considerations about the necessity of being able to represent mathematical notions in at least two different registers. Considerations of different nature had an impact on the development of the new register: (1) taking account of a didactical dimension led to make choices allowing the implementation of tasks of conversion between registers, which seem to be essential for conceptual understanding of mathematical notions (Duval 1993); (2) taking account of users' feedback allowed to make some improvements at the interface level. An example was presented in the previous section; (3) respecting the general principles of the development of Aplusix guarantees the coherence of the system after the introduction of the new register of representation of algebraic expressions. As regards the choices made in the design of the Aplusix tree module, it seems that most of them were made internally, i.e., by the developers themselves, and sometimes even individually, i.e., by one of the developers. Decisions are driven by the fundamental design principles in a way that a coherence of the whole system is preserved. Although it seems that the decisions are taken regardless the school context, both teachers and students are taken into account in the system design. The principles 1 and 3 concern especially students and their interactions with the system. Moreover, the developers are respectful towards the students' ways of editing expressions, which is shown by the decision to make it possible to recover an expression in exactly the same way as the student has edited it, even if the implementation of such a decision was difficult (Trgalova and Chaachoua 2008).

The example of the development of Aplusix illustrates a way the synergy between computer scientists, researchers in math education and users can serve a project of development of educational software.

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QUALITY PROCESS FOR DYNAMIC GEOMETRY RESOURCES: THE INTERGEO PROJECT

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In this contribution, we present the European project Intergeo whose aims are first to develop a common language for a description of geometric figures that will ensure interoperability of the main existing dynamic geometry systems, and second, to gather and to make available pedagogical resources of a good quality. This text focuses on the quality process for dynamic geometry resources aiming at their perpetual improvement.

Keywords: *pedagogical resource, quality of a resource, dynamic geometry, teacher training*

INTRODUCTION

This contribution concerns the issue of integration of ICT tools into teachers' practices and the means of supporting it. One of the keys is to provide teachers with pedagogical resources helping them to develop new activities for their pupils. However, we now know that the availability of resources is not sufficient. On the one hand, the abundance of resources makes difficult to find appropriate and quality resources (Guin and Trouche 2008, Mahé and Noël 2006). On the other hand, the availability of resources does not solve the problem of their appropriation by the teachers, which requires an evolution of teachers' competencies and their conceptions about the role of technology in teaching and learning mathematics (Chaachoua 2004).

This leads to consider the issue of teachers training. Numerous research works pointed out the efficiency of training based on co-design of pedagogical resources (Krainer 2003, Miyakawa and Winsløw 2007). Various training actions have been developed in France based on this principle, e.g., SFODEM and Pairform@nce (Gueudet et al. 2008). In Brazil, AProvaME project aimed to study the effects of a collaborative design of resources involving ICT tools by the teachers on their conceptions about the notion of proof and its teaching, as well as about the role of technology in mathematics learning (Jahn et al. 2007).

THE INTERGEO PROJECT

Despite the availability and accessibility of ICT tools, and despite the recommendations in the curricula to use technology in France and in Brazil, teachers are reluctant to use these technologies (Artigue 2002). In the case of dynamic geometry systems (DGS) several reasons explain this resistance. The most important is certainly the shift in considering mathematical activity and teacher profession caused by the introduction of ICT into mathematics classroom (Lagrange and Hoyles 2006). However, other obstacles to using DGS by the teachers can not be neglected.

First, the complexity of choice of a reliable and easy to use DGS among a number of existing systems, and the resulting constraints on the choice of resources that must match the chosen DGS. Next, it is hard to find pedagogical resources appropriate to a specific educational context. This can be attributed to a great amount of resources available on the Internet, but mostly to the lack of metadata, providing an accurate description of the resource content. Moreover, available resources do not often have the required quality to be used in a classroom. The difficulty for a teacher to evaluate quality and adequacy of a resource to her/his specific context is an obstacle to the ICT integration. For this reason, tools for indexing resources, as well as evaluating their quality appear essential.

These considerations lead to 3 goals of Intergeo project (www.inter2geo.eu/fr): (1) interoperability of the main existing DGS, (2) sharing pedagogical resources, and (3) quality assessment process of resources discussed in this paper.

THEORETICAL BACKGROUND

Notion of pedagogical resource

First, it is important to clarify what we mean by pedagogical resource. Indeed, Noël (2007) points out that the issue of resource evaluation relies on the definition of what is a pedagogical resource. Nevertheless, according to the author, in spite of numerous efforts, the definition of pedagogical resource remains vague and rather broad in its scope. The most often used one is drawn from LOM standards (2002): "*… any entity, digital or non-digital, that may be used for learning, education or training*" (p.5). Flamand (2004) specifies that in order to enhance learning, a Learning Object has to possess intrinsically a pedagogical intention. Thus, for the purposes of Intergeo project, we will consider as resources those "entities" (dynamic geometry figures, texts…) for which pedagogical intention is specified.

In addition, we share Trouche and Guin's (2006) point of view, which, referring to the instrumental approach (Rabardel 1995), considers a pedagogical resource as an artefact that needs to be transformed into an instrument by a teacher in the process of its use in her/his class. For the authors, usage of a resource is a condition for its existence. Resources are therefore living entities in evolution through their usages. In this perspective, the quality assessment process of Intergeo DG resources aims at enabling their perpetual improvement.

Quality assessment process

The quality of a resource depends on its intrinsic characteristics, as well as on its adequacy to the context in which it will be used. A given resource can be "good" in one context and "poor" in another. Thus clarifying its educational goals and the school context in which its use is intended is also essential in determining and improving the quality of the resource.

Mahé and Noël (2006) constituted an evaluation typology based on a detailed analysis of evaluation means set up by various web sites offering pedagogical resources: a priori evaluation by the adherence institution; validation of resource conformity to a deposited content; peer-review by expert teachers; user evaluation; cross-evaluation both by peers and users. The quality assessment in Intergeo project regarding DG resources consists of an evaluation by users and a peer review of a number of resources by a group of teachers supervised by math education researchers based on a priori analysis, use in a class, and a posteriori analysis of the resources. This process corresponds to the 5th type of evaluation mentioned above, rarely encountered according to the authors.

Mahé and Noël (ibid.) bring to light critical aspects of a resource to take into account in the evaluation process: technical aspect, content, design aspect and metadata. Criteria we have set up for the quality assessment process of DG resources draw from these categories, as well as from theoretical frameworks suitable for resource analysis: (1) *didactic theories*, namely Brousseau' theory of didactic situations offering tools for analysing pupil's activity and teacher's role, and Chevallard's anthropological theory allowing to address issues of resource adequacy to institutional expectations, and (2) *instrumental approach* (Rabardel 1995) providing a framework for instrumented activity analysis.

USER EVALUATION OF THE QUALITY OF A RESOURCE

Our elaboration of a questionnaire for DG resource quality evaluation by users started by listing characteristics or elements of a resource related to its mathematical, didactical and pedagogical quality. We attempted to obtain a list as complete as possible. These characteristics were classified into 9 classes considered as relevant indicators of the resource quality: metadata, technical aspect, mathematical dimension of the content, instrumental dimension of the content, potentialities of DG, didactical implementation, pedagogical implementation, integration of the resource into a teaching sequence, usage reports. In what follows, we give an overview of criteria related to four classes referring to mathematical and didactical value of a resource.

Mathematical dimension of the content of a resource

There is no doubt that, for a resource to be usable in a school context, its content has to be mathematically correct. Adequacy of the content with the curricula allows the evaluation of the resource utility. Finally, mathematical activities need to be in adequacy with the declared educational goals.

Criterion	*Question*
Validity	Are the activities in the resource correct from a mathematical point of view?
Adequacy to the curriculum	Are the activities in adequacy with curricular and institutional constraints?
Adequacy to declared goals	Are the activities in adequacy with the declared educational goals?

Table 1. Mathematical dimension of the content of a DG resource

Instrumental dimension of the content of a resource

When a resource includes a DG file, it is necessary to check the coherence between the proposed activity and the geometric figure. In addition, the figure should behave as expected. Particular attention should be paid to the handling of limit cases and of numerical values such as measures of lengths and angles. Indeed, the dynamic diagram should behave according to mathematical theories and didactical goals. If special functionalities, such as macro-constructions, are used, a description of their operating mode will make easier the appropriation of the resource by a teacher.

Criterion	*Question*
Adequacy of diagrams	Do the dynamic diagrams correspond to the proposed activities?
Behaviour of diagrams	Do the dynamic diagrams behave as expected in the activity?
Management of limit cases	Is the management of limit cases in the dynamic diagrams acceptable from the mathematical point of view?
Management of numerical values	Is the management of numerical values acceptable in the sense that it does not hinder mathematical aims of the activity?
Special functionalities	If the diagrams rely on special functionalities (e.g., macro-construction), is their operating mode clearly described?

Table 2. Instrumental dimension of the content of a DG resource

Potentialities of dynamic geometry

Numerous researches on DG put forward its potentialities and their contribution to the learning of geometry (Laborde 2002, Lins 2003, Tapan 2006). Criteria in this class aim first at evaluating how these potentialities are exploited in the resource, and more specifically to what extent DG contributes to improve learning activities comparing to paper and pencil environment. Second, its contribution to the achievement of educational goals is also analysed. This class comprises two criteria: (1) specific features of DG offering an added value to the resource, (2) role and use of drag mode, drawing on diversity of DG potentialities highlighted by research works (Laborde 2002, Healy 2000, Mariotti 2000). Even if a resource cannot benefit from each of them, we consider a resource that does not take any advantage of DG is of poor quality. Our hypothesis is that teachers perceive DG mainly as enabling to drag points to make pupils observing invariant properties (Tapan 2006).

Criterion	*Question*
Elements contributing to the added value of DG in the resource	Is DG a visual amplifier improving graphical quality and accuracy of diagrams?
	Is DG used to obtain easily and quickly many cases of a same figure?
	Does DG provide an experimental field for the learner's activity?
	Do the feedbacks enable students validate their constructions by themselves?
	DG offers a possibility to articulate different representations of a same mathematical problem. Is this possibility used in the resource?
	Does DG allow students to overcome the spatio-graphical characteristics of a diagram to focus on its geometrical properties?
	Is the activity specific to DG, i.e., it would be meaningless without it?
	Does the use of DG in the activity contribute to achieve the educational goals?
role of the drag mode in the	Is dragging used to illustrate a geometrical property, i.e., students are encouraged to drag elements and observe a given property that is invariant while dragging?
	Is dragging used to conjecture geometrical relationships, i.e. the point is to

	observe whether a supposed property is invariant while dragging elements?
	Is dragging used to study different cases of the diagram?
	Is dragging used to obtain a specific configuration satisfying given conditions?
	Is dragging used to identify dependencies between objects?
	Is dragging used to illustrate link between hypotheses and conclusion in a theorem, i.e., the point is to momentarily satisfy hypotheses by dragging elements (soft construction) and consider obtained properties as necessary consequences?
	Is dragging used to explore trajectories of geometrical elements (locus, trace)?
	Is the use of dragging explicitly mentioned in the instructions for students?

Table 3. Potentialities of dynamic geometry

Didactical implementation of the resource

Trouche (2005) points out that a successful integration of ICT requires a specific organization of pupil-computer interactions, which he calls "class orchestration". The author emphasises the importance of instrumental processes management in relation with learning mathematics. For this reason, we are convinced that a quality resource should provide a kind of assistance related to the class orchestration by means of elements concerning mathematics learning management with technology, which would help the teacher organize favourable learning conditions. We propose the criteria and questions, reported in table 4, addressing the issue of didactical implementation of a resource.

Criterion	*Question*
Mathematical learning management	Do the students get involved easily in the proposed activity?
	Does the activity let enough initiative to students to choose their strategies?
	Does the resource describe students' possible strategies and answers?
	Does the resource provide information about teacher reactions to students' errors?
	Does the resource provide information about the teacher interventions at the beginning of the activity with the students?
	Does the resource provide information about the teacher interventions making the students' strategies evolve?
	Does the resource provide information about the teacher interventions during the phase of synthesis?
	Does the resource provide information about the validation phases?
	Does the resource discuss main characteristics of the activity, their effects on students' behaviours and other possible choices?
Instrumented activities management	Does the resource provide information about feedback from the software?
	Do the dynamic diagrams provide feedback enabling the student to progress in solving the given tasks?
	Does the resource provide information about the possible teacher interventions regarding instrumental aspects of the activity?

Table 4. Didactical implementation of a resource

The resulting questionnaire comprises 9 classes with 59 questions altogether. It deals with a great variety of aspects of a quality DG resource and should be comprehensive. However, the questions are not homogenous from the point of view of expertise required to understand and to be able to provide a sound answer to each question. It can be expected that all users will not evaluate all aspects of a resource, but they will rather focus at those that correspond to their own expertise and their own representation of what is a quality resource. Nevertheless, the quality of a resource will take account of all evaluators; therefore we expect that each aspect will be evaluated by some of the users.

Given the length of the questionnaire, it seemed necessary to start by proposing a lighter version to users focusing on a few large questions (one per class) addressing globally each aspect of the resource. At the same time, the user will have the possibility to deepen her/his answer by answering more precise questions related to aspects s/he will wish to analyse further, according to her/his expertise. Moreover, s/he will be given opportunity to go back to the evaluation repeatedly. Note that the process of resource ranking (under development) will take account of the user's declared expertise and assign a weight to each provided answer accordingly.

Since the end-users of the questionnaire are teachers, we wished to test relevance and clarity of the questions. For this purpose, we organized a pilot experimentation with a group of teachers using a simplified version of the questionnaire. The experiment and some results are described in what follows.

EXPERIMENTATION

Some elements of the initial questionnaire available in (Mercat et al. 2008) have been tested in Brazil, within an in-service teacher training "Geometry" module. Our goal was to analyse the relevance of evaluation criteria we defined, as well as to understand what a quality resource is for the teachers. A few more open questions were added aiming at identifying elements of a resource the teachers consider as helpful in order to appropriate and use the resource in their classes. A DG resource has also been designed to control some of its aspects for the experiment purposes and to be relevant for a teacher training.

Presentation of the resource and of the questionnaire

The resource addresses the "quadrilaterals" topic and makes use of Cabri-geometry. It is constituted of a student worksheet, a teacher document and three DG files: two dynamic figures (cf. Fig. 1) and one macro-construction.

The teacher document provides a description of the resource: topic, school level, educational goals, prerequisites and required material. It also provides a brief presentation of the suggested organization of the sessions: classroom setting and roles of teacher and students.

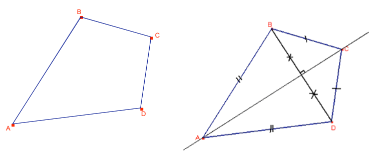

Figure 1. Dynamic figures composing the resource

The first mathematical activity, whose aim is to introduce a special type of a quadrilateral, an isosceles kite, draws from the idea of a "black box" specific to DG environments. It consists in reproducing a geometrical figure that behaves in the same way as a given model. Students are expected to explore the model in order to identify relationships between its elements, then to reconstruct the kite and validate their construction by using the macro-construction. In the resource, the exploration phase is partly guided to lead the students to characterize a kite by means of a maximum of its properties (related to its sides, angles and diagonals). Indeed, the activities are intended for 12-14 year old students and the instructors consider inappropriate to let them completely responsible of exploring the figure and identifying properties and relationships linking its elements. In the second activity, the students are invited to explore the figure and to conjecture a possibility to obtain other types of quadrilaterals (square, rhombus, non squared rectangle) from the kite. In both activities, the drag mode is essential to explore given dynamic diagrams.

For the purpose of the experiment, we selected and adapted several questions from the Intergeo questionnaire (cf. Fig. 2), namely those concerned with mathematical and instrumental quality of the resource, potentialities of DG and didactical implementation of the resource. The questions regarding DG are intentionally open aiming at highlighting which elements the teachers spontaneously mention as contributing to the added-value of DG in the resource.

Regarding the **pupil's worksheet**:

1. Are the texts of the activities:
   a. Clearly formulated?
   b. Mathematically correct?
2. Are mathematics tasks proposed to pupils easily identifiable?
3. Are DG figures provided with the resource:
   a. Easily accessible?
   b. In adequacy with the proposed activities?
   c. Do the figures behave as expected?
4. Would you use this kind of resource with your pupils? With or without modifications? Which ones?

Regarding the **dynamic geometry**:

1. What do you think about the role of dynamic geometry in the resource? Do you think this sequence could be proposed in paper-and-pencil environment?
2. If not, what aspects justify the use of the software?
3. Does the use of drag mode contribute to the achievement of the declared educational goals?

Regarding the **teacher document**:

1. What elements of the teacher documents do you consider as essential and the most relevant for a pedagogical use of the resource?
2. Do you think this document is sufficient for an easy appropriation of the proposed teaching sequence? Do you miss some information? Which?
3. Would you like to have other types of material or documents? Which ones?
4. Suppose that some of your colleagues have tested the resource in their class. What information do you consider important to be shared their experience?

Figure 2. Questionnaire for resource evaluation used in the experiment

Written answers provided by the teachers were one kind of data we gathered. These were completed by field notes of an observer recording relevant elements of exchanges among teachers.

Experimentation and first results

The experimentation consisted in one 2h30 training session for 22 secondary mathematics teachers, who had, in average, six years of experience in teaching and most were "beginners" in DG. The training session was organized in three phases: solving activities from the student worksheet, a priori analysis of these activities, and analysis of the resource guided by the questionnaire (cf. Fig. 2). In what follows, we describe the phase 3 and present the first results.

In the teacher document, the participants particularly appreciated the brief description of the sequence considered as a kind of the resource "*visit card*", as well as the synthetic description of the sequence organisation: "*very well like that, one gets directly every essential information*"; "*one understands immediately how to organise the sequence*".

As regards the student worksheet, the teachers have found the tasks easily identifiable, mathematically correct and clearly formulated. A special attention was paid to the vocabulary with the intention to make the wording of activities accessible to pupils. The teachers used these worksheets also to understand the sequence organisation and its progression: "*student sheets allow us to understand well the whole sequence and to spot contents and objectives"*; "*Student sheets are very well designed. […], one sees clearly the sequence progression: observation of sides, symmetry between vertices and angles. Then, the construction is proposed and finally the study of some cases […]*".

Regarding elements helpful for resource appropriation but missing in the resource, the teachers expressed a need to understand how the macro had been constructed and how it works. They would also have liked to have more information about the teacher's role: what interventions and when, particularly during the institutionalisation phases; how to assist students' work. Some teachers pointed out that a document with reports of use, containing expected solutions and answers, but also possible students' difficulties accompanied with advices how to cope with them (e.g., student worksheet with commentaries for a teacher) would be helpful for a better appropriation of the resource.

Regarding DG, all teachers find unquestionable its contribution in the resource: "*activities specific to Cabri*"; "*the software is essential*"; "*impossible without Cabri*". This is not surprising since the resource was designed for. The teachers state more precisely that "*the software favours checking of properties*"; "*without drag mode and possibility to modify diagrams, properties wouldn't be visualized"*. They spontaneously mention that dragging enables manipulating the figure and thus identifying its properties; checking properties; obtaining easily many different cases of a same figure; constructing figures easily, quickly and more precisely; making conjectures.

It is important to note that the teachers formulated all these criteria spontaneously, but they admitted that they would not have been able to do it without the framework of the questionnaire and without having done previously an a priori analysis of the resource. The questionnaire helped them focus on important aspects of the resource and they were able to provide a deeper analysis than expected. Thereof, the criteria set up for the evaluation questionnaire seem to be understandable by teachers, but what's more, they helped them analyse the quality of the resource. Thus, the questionnaire is not only a tool for characterizing the quality of a resource and for highlighting aspects to be improved, but it can also be used to train users' awareness of positive and negative aspects of a resource and in this way develop their professional skills enabling them to use it efficiently with their pupils.

CONCLUSION

The results from the experimentation show the importance of training teachers to resource analysis. Indeed, the questionnaire helped the teachers focus on important aspects of the resource to look. These aspects were rarely taken into account before the training session. Among those, there is the teacher document containing information about the implementation of the resource and the added value of DG, in particular the role of drag mode.

On the other hand, the quality assessment process will lead to an improvement of a quality of resources, both at the metadata level highlighting information allowing an easier spotting of relevant and quality resources and at the level of the resource itself. Indeed, the quality criteria may be considered as a grid allowing to improve certain aspects of resources or to design new resources satisfying these criteria from the very beginning. Thus, this process can eventually give rise to a model that would act as a guide for resource designers by pointing necessary elements and helping make them explicit in an understandable and accessible way for potential users.

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SYSTEMIC INNOVATIONS OF MATHEMATICS EDUCATION WITH DYNAMIC WORKSHEETS AS CATALYSTS

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With reference to theories of cybernetics the paper proposes a general theoretical framework for initiatives aiming at systemic innovations of educational systems. It shows that it is essential to initiate incremental-evolutionary changes on the meta-level of beliefs and attitudes of the agents involved. For the theoretical foundation of concrete activities in mathematics education the didactic concept of learning environments is developed on the basis of constructivist notions of teaching and learning. Such learning environments may integrate dynamic mathematics for educational processes. So technology and especially dynamic worksheets can be considered as means and catalysts for improvements of mathematics education on system level.

Keywords: *systemic innovation, learning environment, dynamic mathematics*

INNOVATIONS IN COMPLEX SYSTEMS

There are many efforts to innovate educational systems – on regional, national and international levels – aiming at changes of teaching and learning. For understanding the structure of such initiatives a short glance at theories of cybernetics is quite useful.

Innovations

The OECD defines an *innovation* as the implementation of a new or significantly improved product, process or method (OECD, Eurostat, 2005, p. 46). Thus an innovation requires both an *invention* and the *implementation* of the new idea.

In the educational system we are in a situation where lots of concepts, methods and tools have been developed for substantial improvements of teaching and learning. Three examples:

(1) There is a wide range of current pedagogical theories that emphasize self-organised, individual and cooperative inquiry-based learning.

(2) There exists a huge amount of material for teaching and learning in a constructivist manner – available e.g. in electronic data bases or by print media.

(3) A large variety of software and other tools for the integration of ICT in educational processes has been developed.

But for real innovations these promising theories and products have to be implemented in the educational system. Here implementation means a good deal more than diffusion or dissemination of material (papers, guidelines, software tools etc.). And implementation should reach the real agents in the school system, i.e. the teachers and students, their thinking and their working. Let's remember the three examples from above:

(1) Teachers should teach according to current pedagogical concepts.

(2) The proposed new task culture should become standard in everyday lessons.

(3) ICT should be used as a common tool for exploring mathematics.

So for substantial innovations we do not need further material. We need changes in teachers' and students' notions of educational processes, in their attitudes towards mathematics and in their beliefs concerning teaching and learning at school. Hence the crucial question is: How can substantial innovations in the complex system of mathematics education be initiated and maintained successfully?

Complex Systems

In theories of cybernetics a system is called "complex", if it can potentially be in so many states that nobody can cognitively grasp all possible states of the system and all possible transitions between the states (Malik, 1992; Vester, 1999). Examples are the biosphere, a national park, the economic system, mathematics education in Europe and even mathematics education at a concrete school.

Complex systems usually are networks of multiply connected components. One cannot change a component without influencing the character of the whole system. Furthermore real complex systems are in permanent exchange with their environment.

Maybe this characterization of complex systems seems a bit fuzzy. But, nevertheless, it is of considerable meaning. Let us regard the opposite: If a system is not complex, someone can overview all possible states of the system and all transitions between the states. So this person should be able to steer the system as an omnipotent monarch leading it to "good" states. In contrast, complex systems do not allow this way of steering.

Steering of Complex Systems

The fundamental problem of mankind dealing with complex systems is how to manage the complexity, how to steer complex systems successfully and how to find ways to sound states.

With reference to theories of cybernetics two dimensions of steering complex systems can be distinguished (Malik, 1992). The first one concerns the manner, the second one the target level of steering activities (see figure 1).

The method of *analytic-constructive* steering needs a controlling and governing authority that defines objectives for the system and determines ways for reaching the aims. Hierarchical-authoritarian systems are founded on this principle. However, fundamental problems are caused just by the complexity of the system. In complex systems no one has the chance to grasp all possible states of the system cognitively.

So the analytic-constructive approach postulates the availability of information about the system that cannot be reached in reality.

In contrast *incremental-evolutionary* steering is based on the assumption that changes in complex systems result from natural growing and developing processes. The steering activities try to influence these systemic processes. They accept the fact that complex systems cannot be steered entirely in all details and they aim at incremental changes in promising directions. The focus on little steps is essential, since revolutionary changes can have unpredictable consequences which may endanger the soundness or even the existence of the whole system.

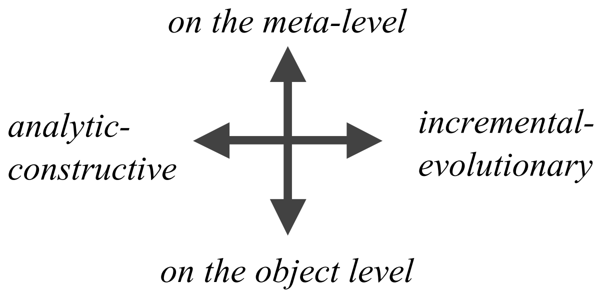

Figure 1: Steering of complex systems

The second dimension distinguishes between the object and the meta-level. The *object level* consists of all concrete objects of the system. In the school system such objects are e.g. teachers, students, books, computers, buildings etc. Changes on the object level take place if new books are bought or if a new computer lab is fitted out. Of course such changes are superficial without reaching the substantial structures of the system.

The *meta-level* comprehends e.g. organizational structures, social relationships, notions of the functions of the system etc. In the school system e.g. notions of the nature of the different subjects and beliefs concerning teaching and learning (e.g. Pehkonen, Törner 1996, Leder, Pehkonen, Törner 2002) are included.

Innovations in Complex Systems

How can substantial innovations in the complex system "mathematics education" be initiated successfully? The theory of cybernetics gives useful hints: Attempts of analytic-constructive steering will fail in the long term, since they ignore the complexity immanent in the system. Changes on the object level do not necessarily cause structural changes of the system. According to the theory of cybernetics it is much more promising to initiate *incremental-evolutionary changes on the meta-level* (see figure 2). They are in accord with the complexity of the system and do not endanger its existence. Nevertheless, they can cause substantial changes within the system by having effects on the meta-level, especially when they work cumulatively.

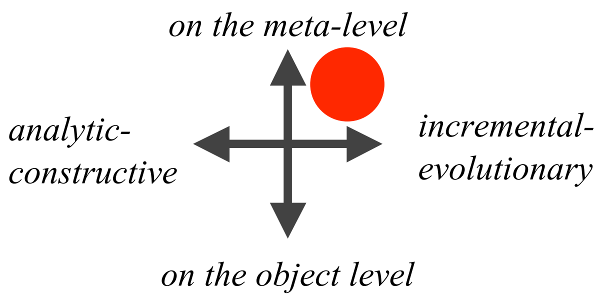

Figure 2: Innovations in complex systems

LEARNING ENVIRONMENTS WITH DYNAMIC WORKSHEETS

Aspects of Learning

Learning is a very complex phenomenon. Initiatives aiming at the development of mathematics education have to take in account the nature of learning. Let us have a very short glance at some fundamental aspects of learning (e.g. Reinmann-Rothmeier & Mandl, 1998; Haberlandt, 1997) which form a background for the latter:

- Learning is a *constructive* process. Knowledge and understanding cannot be simply transported from teachers to students. Cognitive psychology describes learning as a process of construction and modification of cognitive structures. From the view of neurobiology learning is the construction of neuronal networks. Connections between neurons develop and change.
- Learning is an *individual* process. Learning takes place inside the head of each learner. He creates his own knowledge and understanding by interpreting his personal perceptions on the basis of his individual prior knowledge and prior understanding.
- Learning is an *active* process. Cognitive activity means working with the content in mind, viewing it from different perspectives and relating it to the existing network of knowledge.
- Learning is a *self-organized* process. The learner is at least partially responsible for the organization of his individual learning processes. The degree of responsibility may vary in the phases of planning, realizing or reflecting learning processes.
- Learning is a *situative* process. It is influenced by the learning situation. A meaningful context or a pleasant atmosphere can foster learning processes, fear can hamper them.
- Learning is a *social* process. On the one hand the socio-cultural environment has great impact on educational processes. On the other hand learning in school is based on interpersonal cooperation and communication between students and teachers.

Concept of Learning Environments

Considering the aspects of learning noted above the following model seem adequate for teaching and learning processes in school:

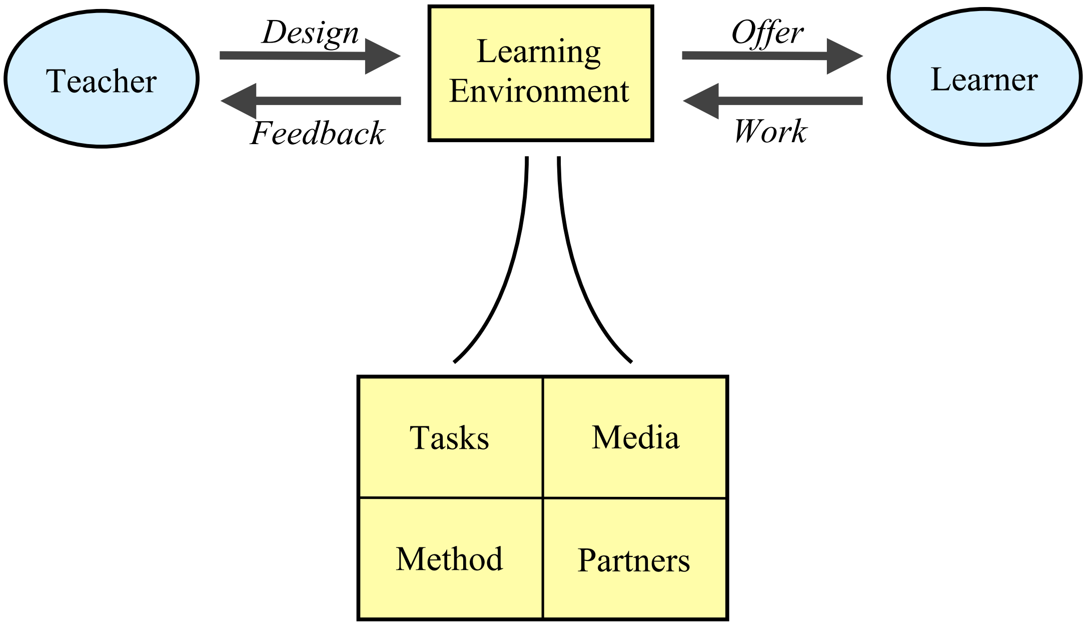

Figure 3: Working with learning environments, four components of learning environments

The *learning environment* is the essential link between the teacher and the learner. This notion includes the *tasks* for the learner's activities, the arrangement of *media* and the *method* for teaching and learning as well as the social situation with the teacher and other learners as *partners* for learning. It belongs to the teacher's field of responsibility to design the learning environment. So he offers a basis for the learner's work. This allows the teacher to get feedback about the learner as well as about the learning environment. This model is based on and extends the didactical concepts of "substantial learning environments" by Wittmann (1995, 2001) or "strong learning environments" by Dubs (1995).

The aspects of learning noted above imply fundamental consequences for the design of learning environments: Tasks should be problem-based with necessary openness for learning by discovery. They should offer meaningful contexts and view situations from multiple perspectives. The teaching methods should make the learners work actively, individually and self-organized. But not less important are the learners' communication and cooperation as well as discussions and presentations of ideas and results. Media can have several supporting functions for these processes.

Before we will discuss the relevance of this model for innovations in educational systems, we look at a specific kind of media which may carry general ideas to practice in school and serve as a catalyst for processes of change.

Dynamic Worksheets

The notion "dynamic mathematics" is currently used for software for dynamic geometry with an integrated computer algebra system, so that geometry, algebra and calculus are connected. When designing learning environments with dynamic mathematics, one faces the necessity to relate dynamic constructions to texts, e.g. for explanations or exercises for the students. For this purpose software for dynamic mathematics – like e.g. GEONExT or Geogebra – can be embedded in HTML-files. So dynamic constructions can be varied on the screen and are combined by the internet browser with texts, pictures, links and other web-elements. This kind of new media for mathematics education is called "dynamic worksheets" (Baptist, 2004; Ehmann, Miller, 2006).

With respect to the model in figure 3 dynamic worksheets are strongly related to all four components of learning environments: Of course they serve as teaching and learning *media*. Since they include text, they may provide *tasks* and instructions for the students. So implicitly they influence the teaching *method* and the cooperation between the learning *partners* (see next section). Hence, when designing learning environments with dynamic worksheets one should carefully take account of all these components and their impact on students' learning.

Figure 4 shows an example: The students are given a mathematical situation leading to an optimization problem. The text is combined with a dynamic construction which helps to understand the context. The rectangular can be moved while fitting exactly in the area between the parabola and the x-axis. The tasks help to structure the lesson according to the methodical concept described in the following section.

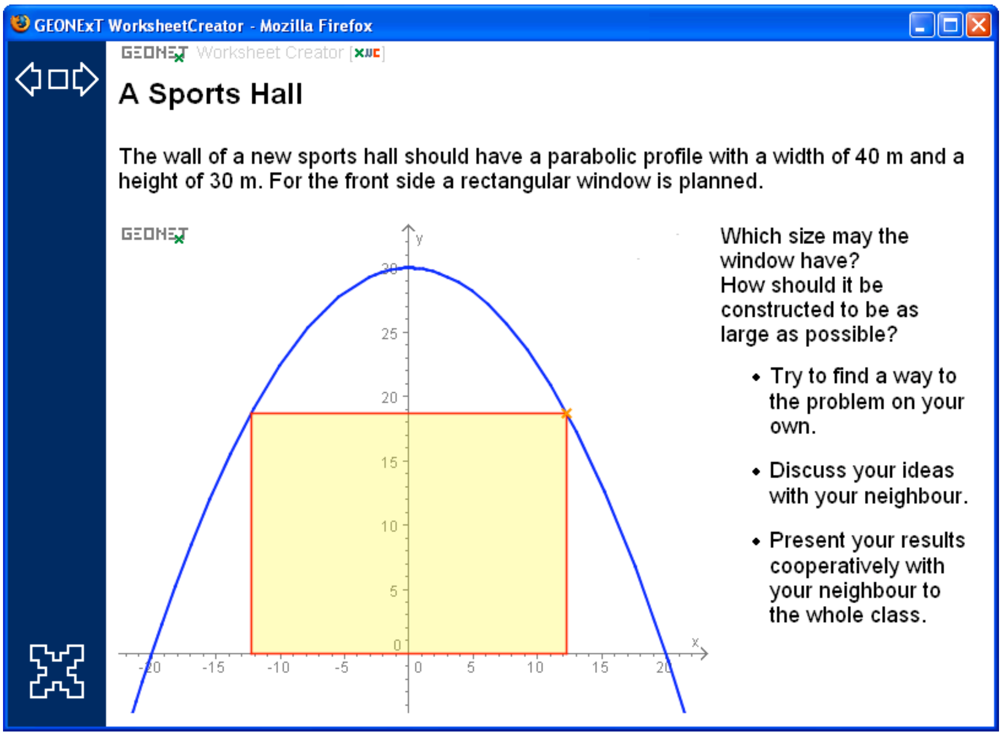

Figure 4: Screenshot of a dynamic worksheet

A Methodical Concept for Learning Environments with Dynamic Worksheets

The use of dynamic worksheets does not automatically improve mathematics education. It is crucial how these media are integrated in teaching and learning processes. If we want to initiate substantial changes on the meta-level of attitudes and beliefs concerning mathematics and mathematics education we have to organize lessons in a way that students work actively, individually, self-organized and cooperatively. They should experience that mathematics is a field for explorations and discoveries. And they should present and discuss their ideas and results cooperatively. Considering the aspects of learning noted above the following four phases structuring lessons with dynamic worksheets methodically are very natural:

1. Individual working: Learning is an individual, active and self-organized process. So at first the students work on their own. They are faced with the necessity to explore the content, to activate their prior knowledge, to develop ideas and to make discoveries. Learning environments with dynamic worksheets offer a framework for such activities and may support them.
2. Cooperation with partners: Learning is a social process. It is very natural that the students discuss their ideas with partners in small groups and that they work on the problems cooperatively. This communication helps to order thoughts and to get further ideas. Meanwhile the teacher may remain in the background or turn his attention to individuals.
3. Presentation of ideas: After having worked individually and in groups the students present their ideas and discuss them in the plenum. The different contributions reveal multiple aspects of the topic and help to view it from varying perspectives. Moreover the students train debating and presentation techniques.
4. Summary of results: Finally the students' results are summarized and possibly extended by the teacher. It is his task to introduce mathematical conventions and to consider curricular regulations. But since the students have already discovered the new content on their own paths, they can more likely integrate the teacher's explanations into their individual cognitive structures.

Table 1: Methodical concept

This methodical concept combines individual learning with working in small groups as well as in the plenum of the class in a very natural way. It is in close relationship to the methodical concepts "Think – Pair – Share" by Lyman (1981) or "I – You – We" by Gallin and Ruf (1998).

Learning by Writing: The Study Journal

The call for papers for working group 7 at CERME 6 emphasizes that technology in school should be considered within a wider range of resources for teaching and learning. Students should draw on ICT in combination with more traditional tools. Accordingly, dynamic worksheets are only one element of rich learning environments. Especially pencil and paper do not lose relevance when student work with the computer. Noting down thoughts helps to order and arrange thoughts. Writing helps to develop understanding for new subject matters. Hence, when using dynamic worksheets students should regularly be asked to draw figures in their exercise book and to write down observations, conjectures, argumentations and personal statements. The exercise book gets the character of a personal "study journal" that accompanies students on their individual learning paths (Gallin, Ruf, 1998).

When designing dynamic worksheets for students' self-responsible learning, one should be aware of the risk that students play with the media as with a computer game quite superficially and do not get to the deeper mathematical content. The regular request of working in the exercise book decelerates the process of clicking through the learning environment. So the students are forced to take their time which is indispensable for individual learning.

Finally, the notes in the study journal ensure that ideas and results are still available when the computer is switched off. They are a basis for further presentations, discussions and summaries in the plenum of class (Baptist, 2004).

INCREMENTAL-EVOLUTIONARY SYSTEMIC INNOVATIONS WITH DYNAMIC WORKSHEETS AS PARTS OF LEARNING ENVIRONMENTS

In their plenary talks at CERME 5 K. Ruthven and M. Artigue observed that current results of activities integrating ICT in school are rather disappointing on system level.

> "Advocacy for new technology is part of a wider reform pattern which has had limited success in changing well established structures of schooling." (Ruthven, 2007) "From the very beginning, digital technologies have been considered as a tool for educational change […]. Unfortunately, the results are far from being those expected" (Artigue, 2007).

For substantial innovations in the educational system there is no lack of general ideas, pedagogical concepts or didactic tools – as discussed above. But there is a wide gap between theoretical knowledge and practice in school. So we have to develop strategies to bridge this gap.

Conclusion: A Pattern for Innovation Projects

Combining the theory of cybernetics and the concept of learning environments using dynamic worksheets we get a pragmatic, but also theory-based way of initiating innovations in school. Activities are most promising, if they focus on incremental-evolutionary changes on the meta-level of beliefs and attitudes of all agents involved. Learning environments with dynamic worksheets may serve as framework for learning processes of teachers *and* students. How can this be done concretely?

As a conclusion from all reflections above we sketch and propose a pattern for innovation projects for mathematics education. (It is realized e.g. by the current project "InnoMathEd – Innovations in Mathematics Education on European Level", see http://innomathed.eu).

(1) The key persons for innovations in school are the teachers. Their beliefs, motivation and abilities are crucial for everyday teaching and learning in school. So regional networks of schools are established which form frameworks for teachers' cooperative learning, exchange of experience and professional development.

(2) Universities are innovation centres for teacher education. They lead the school networks and provide regular and systematic in-service teacher education offers. This teaching and learning is designed according to the aspects of learning and the concept of learning environments described above. So the teachers get acquainted with these theories and concepts by making personal experiences in learning environments designed for them.

(3) Participating schools concentrate on one or a few areas of innovation, e.g. autonomous learning with dynamic worksheets, promoting student cooperation with dynamic worksheets or fostering key competences with dynamic worksheets. It is not promising to aim at total changes of mathematics education – because of the complexity of the system. However, such bounded fields of activity allow teachers to begin with substantial changes without the risk of losing their professional competence in class.

(4) The teachers get acquainted with general ideas and theories of teaching and learning as well as with techniques for constructing learning environments. To bridge the gap between theory and practice the teachers' project activities are strongly related to their regular work at school. They develop learning environments for their students, they use, test and evaluate them in their classes and finally optimize them on the basis of all experiences. In this process they get guidance and coaching by the University leading the network.

(5) All learning environments which are tested, evaluated and optimized are collected in a data based and made available for public use.

(6) Teachers are given possibilities to exchange experiences with colleagues and to participate in teacher education offers on national and international level. Thus they understand that problems and necessities for development have systemic character and concern the fundaments of mathematics education far beyond their own professional sphere. Moreover, they get ideas for innovation activities from a large community.

(7) Finally, further networks of teachers and schools are essential means for dissemination processes in the long term. Experienced teachers coach colleagues from schools starting with innovation activities.

This approach may be called "theory based and material driven". On the basis of the theory of cybernetics and the theories of learning the teachers involved make incremental-evolutionary steps on the meta-level of beliefs and attitudes by designing and working with concrete learning environments for their classes.

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THE LONG-TERM PROJECT "INTEGRATION OF SYMBOLIC CALCULATOR IN MATHEMATICS LESSONS" – THE CASE OF CALCULUS

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A long term project (2003 – 2011) was started to test the use of symbolic calculators (SC) in grammar schools in Bavaria (Germany). The project was firstly done in grade 10. During the 2006/07 school year the project was implemented in grade 11. 732 students at 10 Bavarian grammar schools took part in an empirical investigation. The content taught was calculus: basic properties of functions, limits, continuity, derivatives, and applications of calculus. The evaluation of the project was intended give answers to the following questions: how basic mathematical skills (algebraic transformations, solving equations) changed; how the students used the SC, how they evaluated the use of the new tool. This article presents the results of this project for school year 2006/07.

1. BACKGROUND

In the past, many empirical investigations concerning the use of CAS or symbolic calculators (with CAS) in mathematics teaching have been published (see Guin, Ruthven and Trouche, 2005). The central results of these projects have meanwhile been confirmed by other investigations world wide. The use of a CAS brings a greater meaning to work with diagrams, reinforces experimental work, in which the assumptions were obtained through systematic testing and CAS appears to bring an increase in computer cooperative forms of work. The effects are primarily long term. It is therefore necessary to develop a namely educational concept to evaluate the changes in knowledge and abilities over a longer time period. However, many investigations in this area restrict themselves to the applications of the computer over "just" a few weeks (Schneider, 2000, Drijvers, 2003, Pierce and Stacey, 2004 and Guin et al, 2005) and do not show the long-term effects on the knowledge and ability of the students.

In the school year 2003/04 we started a long term project to test the use of symbolic calculators (SC) – the TI-Voyage 200 and the TI-Nspire – in grammar schools (Gymnasien) in Bavaria (Germany). The project was done in grade 10 and has been repeated in the following two school years with a greater number of classes and with – concerning the use of new technologies – inexperienced teachers. An overview of the empirical investigation and especially of the theoretical background of this project gives Weigand (2008). On account of the positive results of this project, the Bavarian Ministry decided to continue the project. The follow-up project was started in September 2006.

2. THE TEACHING PROJECT – GRADE 11

2.1 The participants

During the 2006/07 school year the project was implemented in grade 11. A total of 732 students at 10 Bavarian grammar schools took part in this project. 412 students in 16 classes acted as the "pilot classes", working with Voyage 200 and/or TI-Nspire. Schools could apply for the participation in the project. The pilot schools have been chosen by the Bavarian Ministry. They are spread over the state. In addition, 320 participants from 11 classes – from the same schools as the pilot classes – formed a "control group" for the purposes of quantitative statistical investigation. The students had different previous experiences; some students had been exposed to the SC in the previous grade 10, but other students came into contact with these systems for the first time during this project.

2.2 The teachers

The project was mainly taught by teachers with little experience of tuition using computer algebra systems (CAS). The project teachers held two three-day meetings during which examples of possibilities and opportunities for SC use were discussed. The teachers jointly prepared a number of suggestions for a range of teaching units intended to highlight the possibilities of using SCs; during the year, the teachers were offered additional learning units[1] by the coordinator (Ewald Bichler). However, there was no uniform overall concept according to which teaching was to be organised in all classes. The personal experience, attitudes and circumstances at the individual schools were too different for this to be possible.

2.3 The learning contents

In grade 11, calculus is taught (in Germany). The content taught was subdivided into the following:

- basic properties of functions (symmetry, monotonicity, variations in function terms and their impact on graphs, …)
- limits, continuity
- differentiability, derivation rules, derivation function(s)
- applications of differential calculus ("classical" functions discussion, extreme value problems)

2.4 Teaching methods with the SC

During the meetings with the teachers at the beginning and in the middle of the school year a theoretical frame of the use of the SC in the classroom was discussed with the teachers. Especially a short insight into the theory of instrumentation was presented and explained with examples (Artigue 2002, Trouche 2005).

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[1] One sort of learning units developed during the project is called "Minute Made Math", more information on www.minute-made-math.com

Concerning the integration of the SC into the problem solving process we distinguished using the SC

- in the beginning of the problem solving process or a concept formation process (the SC as a “discoverer”),
- in the middle of the process (the SC as “solver”) and
- at the end of the process (the SC as a “controller”).

We also emphasized the “rule of three” while working with representations: If possible a problem or the solution of the problem should be represented on a symbolic, graphic and numeric level.

2.5 Research questions:

In the following we concentrate on a selection of the research questions (RQ) of the project:

RQ1. Can any differences be ascertained in terms of core mathematical abilities (substitutions, interpretation of graphs, solving equations, working with tables, and working with formulae) between the pilot and the control groups after one year?

RQ2. Can different effects of SC use be ascertained with “good”, “average” and “weak” students?[2]

RQ3. To what extent have students mastered the SC at the end of the year?

RQ4. In which phases of a problem solving activity do the students use the SC?

2.6 Test instruments

For the purpose of answering the 1st and 2nd questions we took a (classical) pre- and post-test-design – the tests using paper and pencil but no calculator – in pilot and control classes.[3]

For the purpose of answering the 3rd and 4th questions the pilot classes took a test using a SC in February 2007 and June 2007 in which they were asked to record their working methods with the SC in a questionnaire which they completed immediately *after* the test.

3. EVALUATION OF PRE- AND POST-TESTS

3.1 The questions

The pre- and post-test-questions (PP-questions) can be divided into the following groups:

- Questions 1 and 2: doing “classical” simplification of terms

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[2] The performance criteria used relate to the results of the pre-tests at the beginning of the school year.

[3] See: www.dmuw.de/weigand/2009/CERME6/

- Question 4 and 5: solving equations
- Question 5: understanding the concept of root functions
- Questions 6 – 8: seeing the correlation between graph and term
- Question 9: interpreting graphs

3.2 Comparison of results of pre- and post-tests

The post-test was the same as the pre-test. In the following diagram, the *differences* between the average scores achieved for each question in the pre- and post- tests for the pilot and the control group are shown. The "average performance increase" is therefore measured for each question.

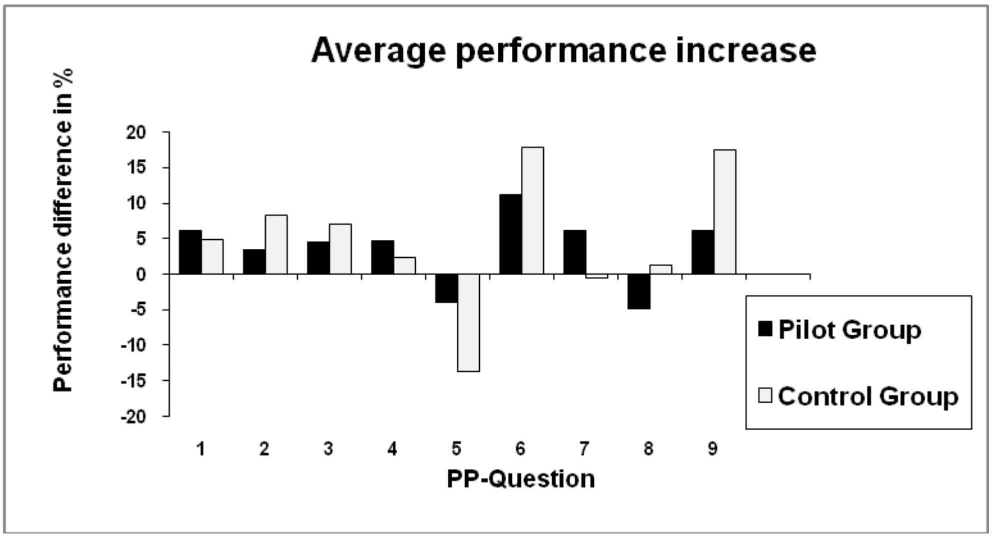

Figure 1: Average performance increase of the pilot and the control group

In PP-questions 5 and 7 the pilot classes' results are significantly better than than those of the control groups (t-Test: PP 5: 0.01, PP 7: 0.02). However, in PP-questions 6 and 9 they are significantly worse (t-Test: PP 6: 0.01, PP 9: 0.01).

Overall there is not a significant difference in the average performance increase between the pilot and control classes. For the comparatively worse result of the pilot classes compared with the control classes (especially for questions PP 6 and PP 9), there are two possible hypotheses. On the one hand it could be due to the fact that the students in the pilot classes were no longer adequately challenged or motivated to tackle this type of "traditional" question with enthusiasm, as they had tackled much more interesting questions during lessons – due to the SC. On the other hand the poor results of the pilot classes when determining functional equations from specified graphs (question 6) could be due to the fact that the students in the pilot classes had seen a large number of graphs – compared with the control group – during the course of the year and were therefore overtaxed by the diversity. However, the students in the control class have probably worked more often with the sine function graph which had been introduced in grade 10.

If, however, the range of performance increases is considered, an interesting picture emerges.

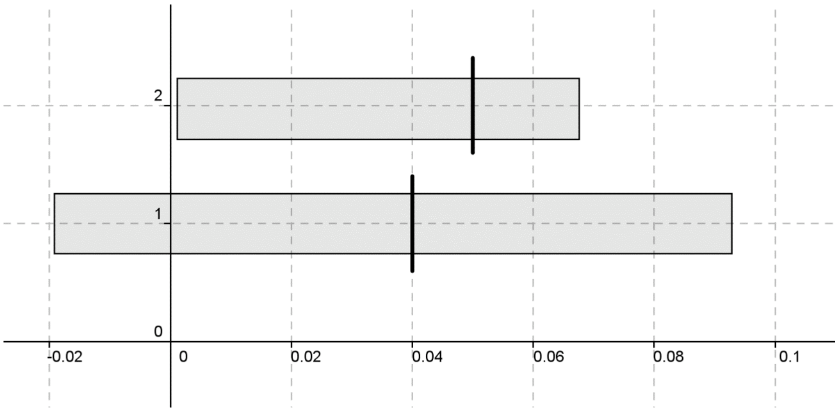

Figure 2: Average value and range of average performance increases in pilot (1) and control groups (2)

With an almost identical average value, it becomes apparent that the differences in performance are more varied with the students in the pilot classes than with the students in the control groups. Therefore, there are students in the pilot classes who benefit more from SC use than students in the control classes. However, there are also students whose results deteriorate compared with the initial test.

The test results can also be interpreted in a positive way for the pilot classes, as there are no differences in terms of classical technical and manual abilities and skills. However, this investigation has deflated hopes that the ability to interpret graphs and transfer between different forms of representation are automatically improved by the use of the SC.

3.3 Scores for "good", "average" and "weak" test participants

In accordance with the results of the pre-test, we divided the test participants into "weak", "average" and "good".[4] The following result is produced when the performances of these groups are compared in terms of pre- and post tests.

[4] The "good" students form the upper performance quartile, the "weak" students the lower performance quartile, and the "average" students are represented by the two central performance quartiles.

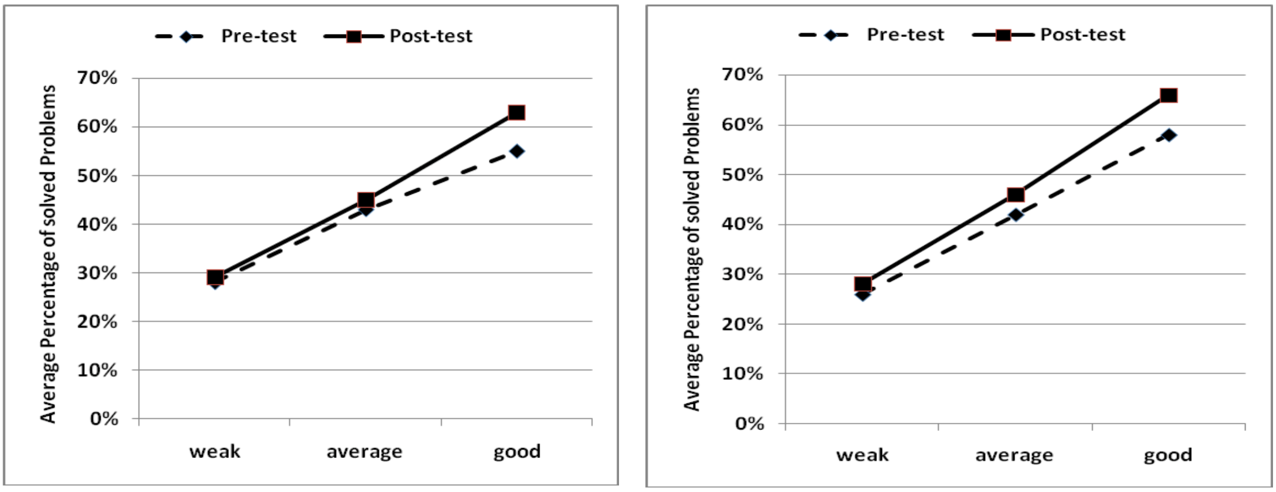

Figure 3: Performances of the pilot group (left) and the control group (right)

Compared with tests carried out in recent years in grade 10 (see Weigand 2008), different behaviour was demonstrated here. Whilst the "weak" students achieved a greater performance increase than the "average" and "good" students in grade 10, the "good" students – both in the control and pilot groups – improved more markedly (by 8 percentage points) than the "average" and "weak" students (by 3 percentage points and 1-2 percentage points respectively) in the grade 11 test.

The differences between the "weak" and "good" groups can be found in the understanding of concepts (question 5) and the transfer between different forms of representation (between graph and equation - questions 8 and 9)). The lack of performance increase in the case of weak students is attributable to the greater cognitive challenges posed by calculus, which may have taken some students to the limits of their capacities so that they were no longer able to follow lessons ("dropout effect").

4. THE SYMBOLIC-CALCULATOR-TESTS (SC-TESTS)

4.1 Research questions

In February and in June the pilot classes took a test where they were allowed to use the SC. Use of the SC was optional for the students, i.e. they decided themselves whether or not they would use the calculator. The two tests consisted of four questions each.[5] In order to establish how calculators were used, we applied a new investigation method: the students completed a questionnaire on SC-use immediately *after the test*, giving details of whether and how they used the calculator. This test was intended to answer the following questions:

1 How do students use the calculator?

2 In which phases of a problem solving process do the students use the calculator?

3 Which functionalities (symbolic – graphic – numerical) do the students use?

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[5] See: www.dmuw.de/weigand/2009/CERME6/

In addition, the teachers were presented with a questionnaire regarding the questions immediately *before the test*, in which they were intended to provide details of the difficulties expected in terms of the questions.

In the following, only a few spotlights of the results will be given.

4.2 Actual use of the SC

The following diagrams show how many students used the SC during the tests in February and in June – according to their own statements:

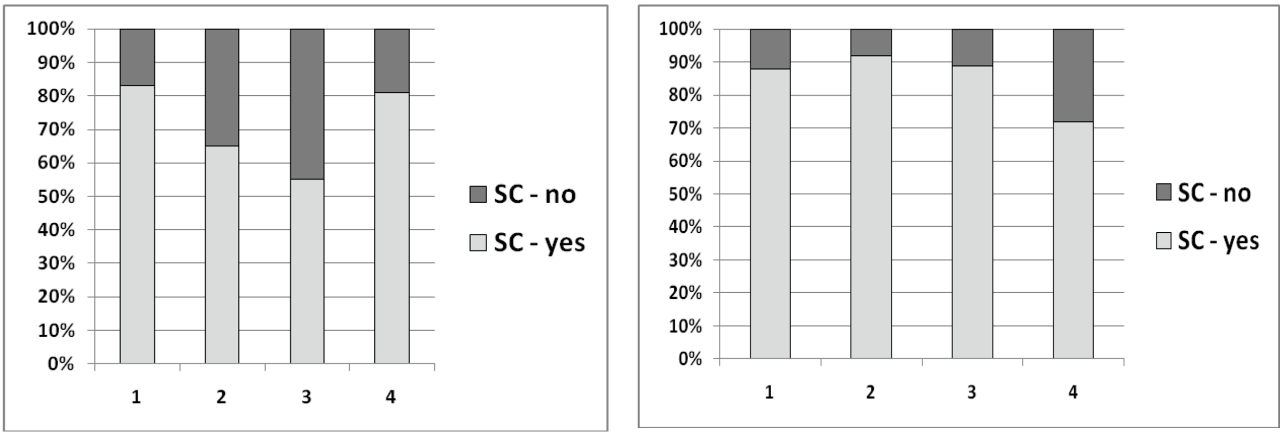

Figure 4: Results of the SC-test in February (left) and June (right) 2007

The difference between SC use in February and in June shows an increase in use of the calculator. More over, those students who used the SC in June when solving the questions scored significantly better than those who did not use it. We attribute this to the fact that it takes a full school year for students to acquire adequate confidence in the SC, as well as knowledge of the benefits of its use as a tool when solving problems, to be able to use these for the purpose of solving problems.

4.3 The SC-use during the problem solving process

The students also provided information in the questionnaire as to whether they used the SC in the beginning, during or at the end of the problem solving process.

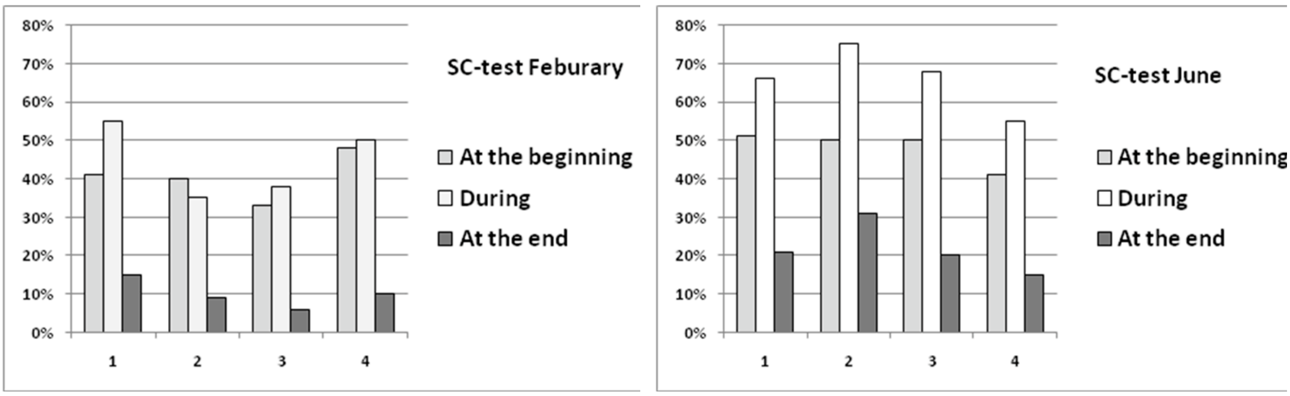

Figure 5: Use of the SC during the course of the solving process (according to

statements made by the students themselves)

When students integrate the SC into their solving process, it is predominantly used at the beginning and during the solving process. If we compare the middle of the school year with the end, we can observe a clear increase in the frequency of positive responses to “during”. This allows us to conclude that the SC is more strongly integrated into the solving process by the students at the end of the school year. A slight increase can also be observed “at the end”, which makes us aware that the use of checking the solution is gaining in importance.

We also asked the students which representations they used while solving a problem with the SC. It appears that the students mainly use the symbolic and graphic possibilities of the SC. Numeric use is very limited. More over they are not familiar with the special advantages or diadvantages of the representations nor do they use the relationship between the different representations. The type of the used representation depends on the one hand very strongly on the way problems are given to the students. If it is asked for a “solution of an equation”, they mainly work on a symbolic level, if it is asked for an “intersection point of two graphs” they work on a graphic level. This shows that the SC is used in a very mechanical way, guided not by the type of problem but by the expressions used in the problem. On the other hand the type of use depends also very strongly on the classes and indicates the significance of the teacher and his or her didactic approach.

4.4 Teachers' predictions

Before each test was carried out, the teachers provided an assessment of the extent to which students would solve the problems. The question has been: “For each problem, a student gets 100 % of the marks for a completely right answer. What do you suggest will be the average score of marks your class gets for problem 1 (2, 3, 4)?” The results are as follows:

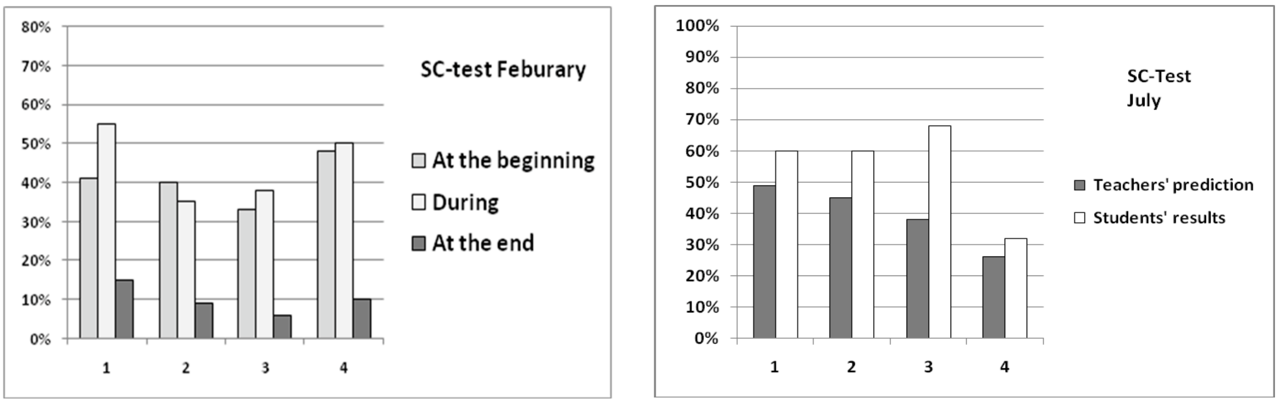

Figure 6: Comparison of teachers' predicted and student results in the SC tests

It is noticeable that the teachers underestimated the students in the June test.

5. Questions for the future

If we summarise the core results of this one-year school project there are some questions for up-coming investigations.

- **Methodology of pre- and post-tests**.

Hopes have not been fulfilled that students in the pilot classes would improve to a greater degree in terms of dealing with and interpreting graphs than students in the control classes. The hypothesis is that students in the pilot classes are not have been adequately challenged or motivated as the result of the largely traditional nature of the test problems. This raises the question whether the used pre- and post-test methodology is an adequate method to answer this question.

- **Polarisation**.

When working with new technologies, polarisation occurs in that some students benefit greatly from SC use, whereas for other students, SC use inhibits performance or even decreases performance. Two thirds of students are of the opinion that the SC was helpful and made them more secure and they classify lessons as “interesting”. Approximately one third of students do not share this view. Are there ways to get all students convinced of the benefits of the SC?

- **Calculator use**.

The reasons for non-use of the calculator are on the one hand the uncertainty of students regarding technical handling of the unit and on the other hand a lack of knowledge regarding use of the unit in a way which is appropriate for the particular problem. Is there a correlation between these two aspects?

- **Period of adjustment**.

The responses of the students confirm that familiarity with the new tool requires a very long process of getting used to it. It is surprising that it took almost a year to establish familiarity with this tool for students to use it in an adequate way. After one year of SC use, confidence in and familiarity with the SC grow. However there is still a large group of students who experience technical difficulties when operating the SC. Will there be ways to shorten this period of adjustment?

- **Solution documentation.**

Students have problems how to record solutions when using the SC. Difficulties with the type and manner in which to document the solution decreased over the year, but still remain at a high level. This latter point will continue to be a permanent challenge when working with the SC, as there is no algorithmic solution for the procedure. Are there documentation rules for all or a special type of problems?

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